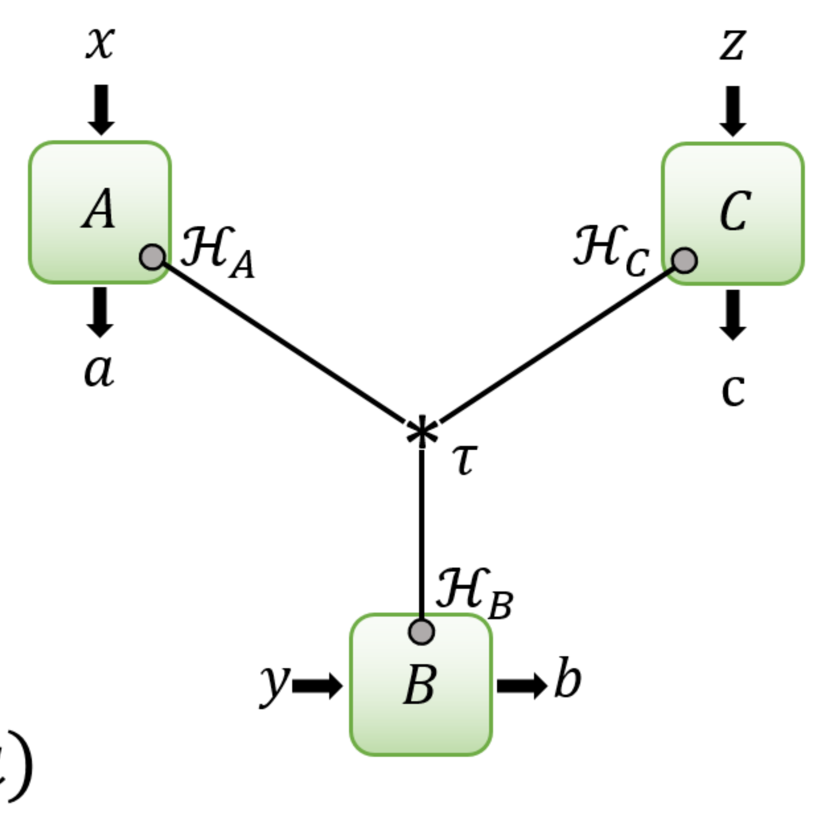
Two convergent NPA-like hierarchies for the quantum bilocal scenario

Marc-Olivier Renou[1,2], Xiangling Xu[1], Laurens T. Ligthart[3]

[1]Inria Paris-Saclay, Bâtiment Alan Turing, Palaiseau, France
[2]CPHT, Ecole Polytechnique, Institut Polytechnique de Paris, Palaiseau, France
[3]Institute for Theoretical Physics, University of Cologne, Cologne, Germany

Known:

- Characterization of Bell scenarios with NPA hierarchy
- Foundational understanding of quantum correlations
- Device-independent quantum key distribution/cryptography



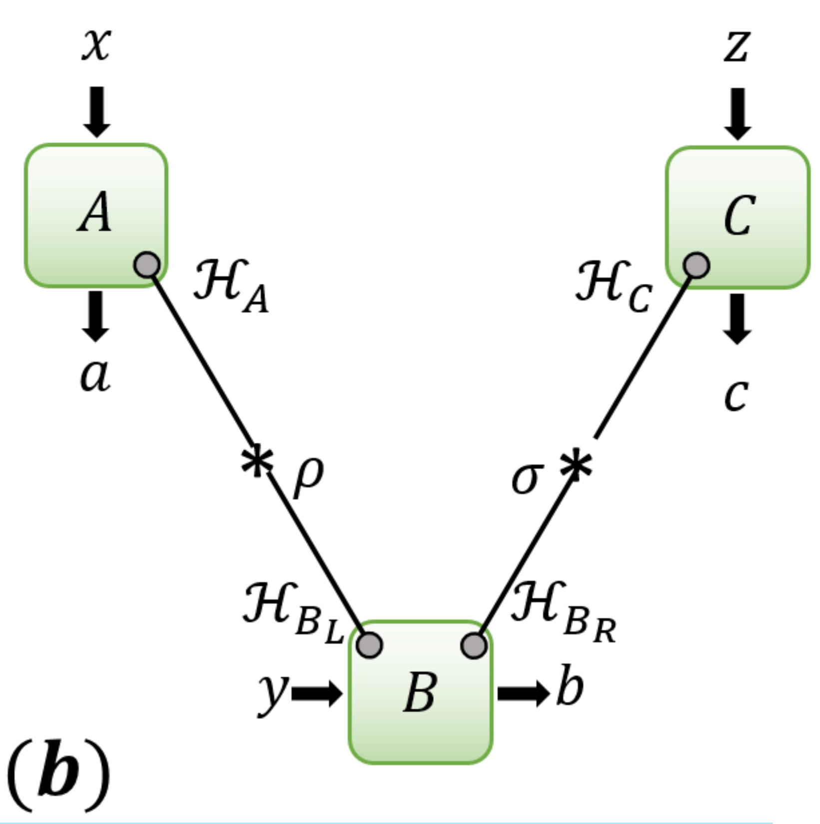
(a)

Tripartite Bell scenario C_{qa}

- Hilbert space $H = H_A \otimes H_B \otimes H_C$ with a shared state τ
- PVMs $\{A_{a|x}\}, \{B_{b|y}\}, \{C_{c|z}\}$
- Born's rule: $p(abc\,|\,xyz) = \mathrm{Tr}(\tau(A_{a|x} \otimes B_{b|y} \otimes C_{c|z})) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z})$
- Certify if a given $\overrightarrow{P} = \{p(abc\,|\,xyz)\}$ comes from Bell scenario?

Our works:

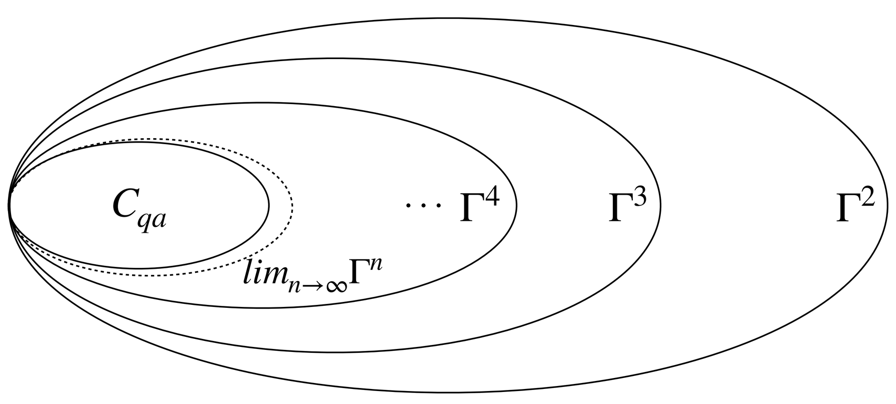
- Network scenarios, multiple independent quantum sources
- NPA hierarchy no longer suffices
- We find two generalizations that can characterize bilocal networks.

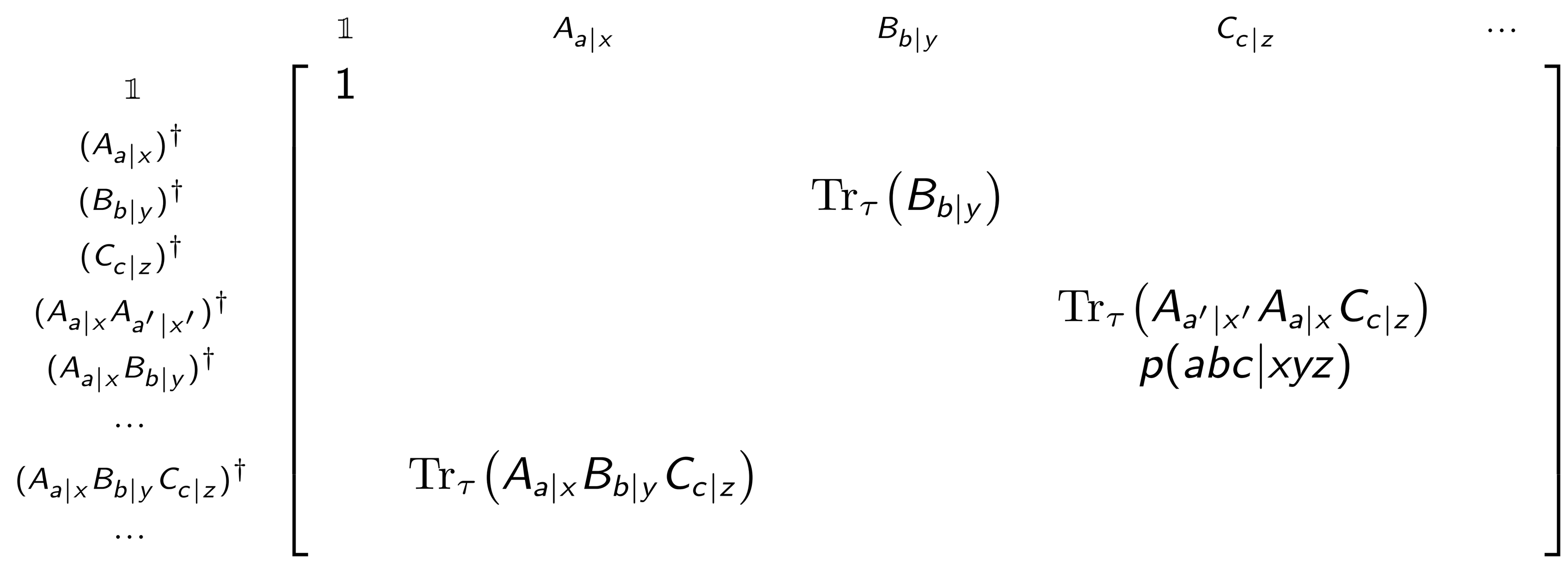


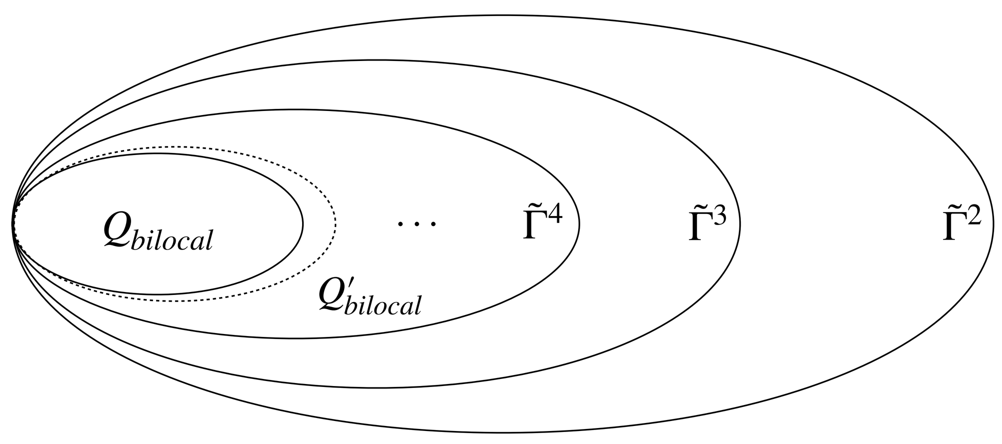
(b)

Bilocal network scenario $Q_{bilocal}$

- $H = H_A \otimes H_{B_L} \otimes H_{B_R} \otimes H_C$, $\tau = \rho_{AB_L} \otimes \sigma_{B_RC}$
- PVMs $\{A_{a|x}\}, \{B_{b|y}\}, \{C_{c|z}\}$, s.t. $[A_{a|x}, \sigma] = [\rho, C_{c|z}] = 0$
- Alice and Charlie are independent: $\mathrm{Tr}_\tau(A_{a|x}C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x})\mathrm{Tr}_\tau(C_{c|z})$, also products of $A_{a|x}$, $C_{c|z}$.
- Simplest network, but $Q_{bilocal} \subsetneq C_{qa}$, so the standard NPA is not applicable. Not even convex!







NPA hierarchy: outer approximation

- Hierarchy of necessary conditions $\Gamma^n, n \geq 2$, such that $\overrightarrow{P} \in C_{qa} \implies \cdots \implies \Gamma^4 \implies \Gamma^3 \implies \Gamma^2$
- Equivalently, if for some n, Γ^n is not satisfied, then $\overrightarrow{P} \notin C_{qa}$
- Moment matrices, defined such as $\Gamma^n_{B_{b|y},B_{b|y}} = \mathrm{Tr}_\tau(B^\dagger_{b|y}B_{b|y}) = \mathrm{Tr}_\tau(B_{b|y}) = \mathrm{Tr}_\tau(\mathrm{Id}^\dagger \cdot B_{b|y}) = \Gamma^n_{1,B_{b|y}}$
- SDP (solvable by computers)

Factorisation bilocal hierarchy $\tilde{\Gamma}^n$

- $\tilde{\Gamma}^n = \Gamma^n$ + factorisation constraints
- Almost the same as the standard Γ^n
- But additional factorization constraints: $\tilde{\Gamma}^n_{A_{a|x},C_{c|z}} = \mathrm{Tr}_\tau(A_{a|x}C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x})\mathrm{Tr}_\tau(C_{c|z}) = \tilde{\Gamma}^n_{A_{a|x},1} \cdot \tilde{\Gamma}^n_{1,C_{c|z}}$
- Not SDP!

Convergence of standard NPA:

If $\overrightarrow{P}$ admits Γ^n for all $n \to \infty$, then $\overrightarrow{P} \in C_{qc}$ is the ***Commutator quantum distribution***

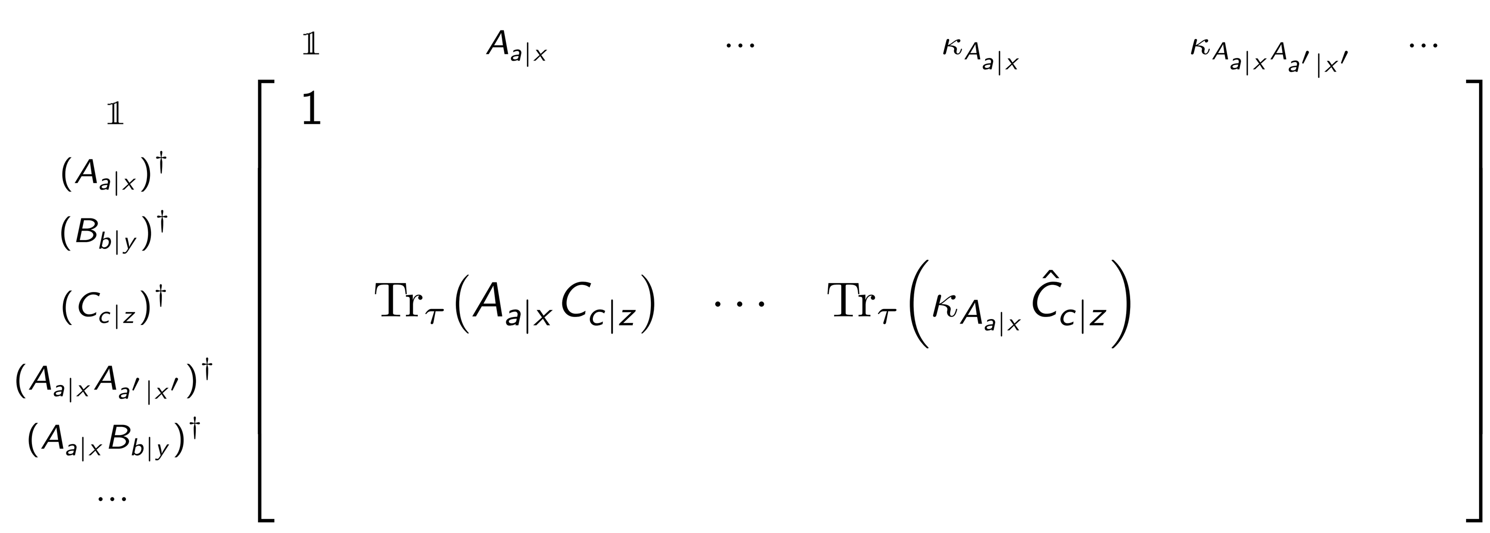
1. A global Hilbert space H with pure density operator τ
2. PVMs $\{A_{a|x}\}, \{B_{b|y}\}, \{C_{c|z}\}$ mutually commute
3. $p(abc\,|\,xyz) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z})$

- Tsirelson's problem: $C_{qa} \subsetneq C_{qc}$, but coincides in finite dimension

Main result 1: convergence of factorization NPA:

If $\overrightarrow{P}$ admits $\tilde{\Gamma}^n$ for all $n \to \infty$, then $\overrightarrow{P} \in Q'_{bilocal}$ is the ***Projector bilocal quantum distribution***

1. *Commutator distribution and*
2. *Projectors ρ, σ on H such that $\tau = \rho \cdot \sigma = \sigma \cdot \rho$*
3. $[A_{a|x}, \sigma] = [\rho, C_{c|z}] = 0$



Main result 2: Convergence of Scalar extension hierarchy

- To make it SDP, new commutative variables: $\kappa_{A_{a|x}}, \kappa_{A_{a|x}B_{b|y}}, \kappa_{A_{a|x}A_{a'|x}C_{c|z}}$ …, + complicated constraints
- Also characterizes $Q'_{bilocal}$

Conclusion and outlook

- We introduced two convergent hierarchies, one of them is SDP, checkable by computers. With [Ligthart&Gross, 2023] and [Klep et al., 2023], bilocal scenario is completely characterized in C^*-algebraic/Heisenberg picture. Next step is to look at more general networks, such as triangle, with possibly quantum inflation technique.