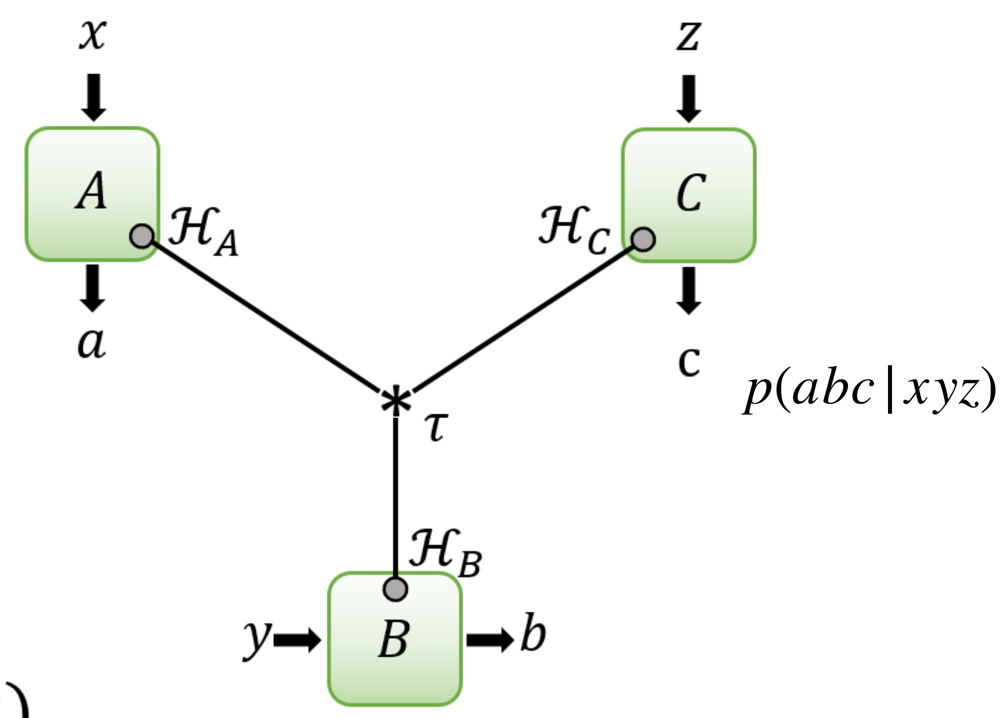
Characterizing quantum bilocal network scenario with generalized NPA hierarchies

Based on arXiv:2210.09065 [Renou, Xu, Ligthard, 2022]

Xiangling Xu, Inria Saclay Île-de-France

Tripartite Bell scenario: standard QM C_{qa}



(a)

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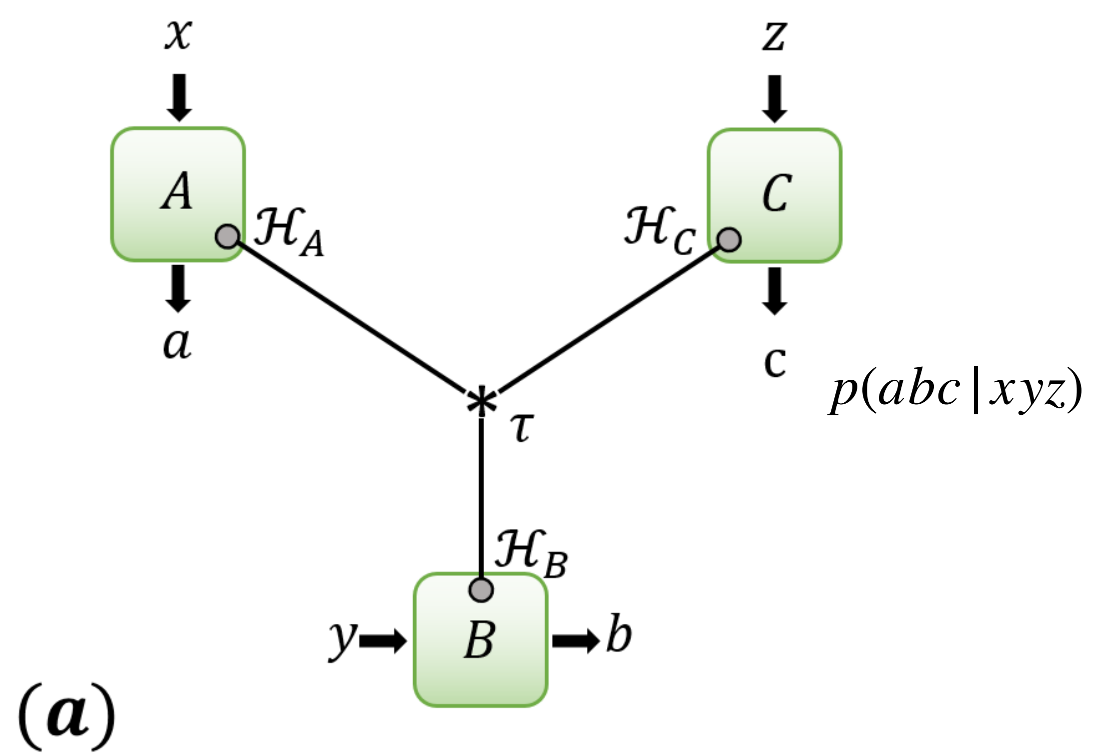

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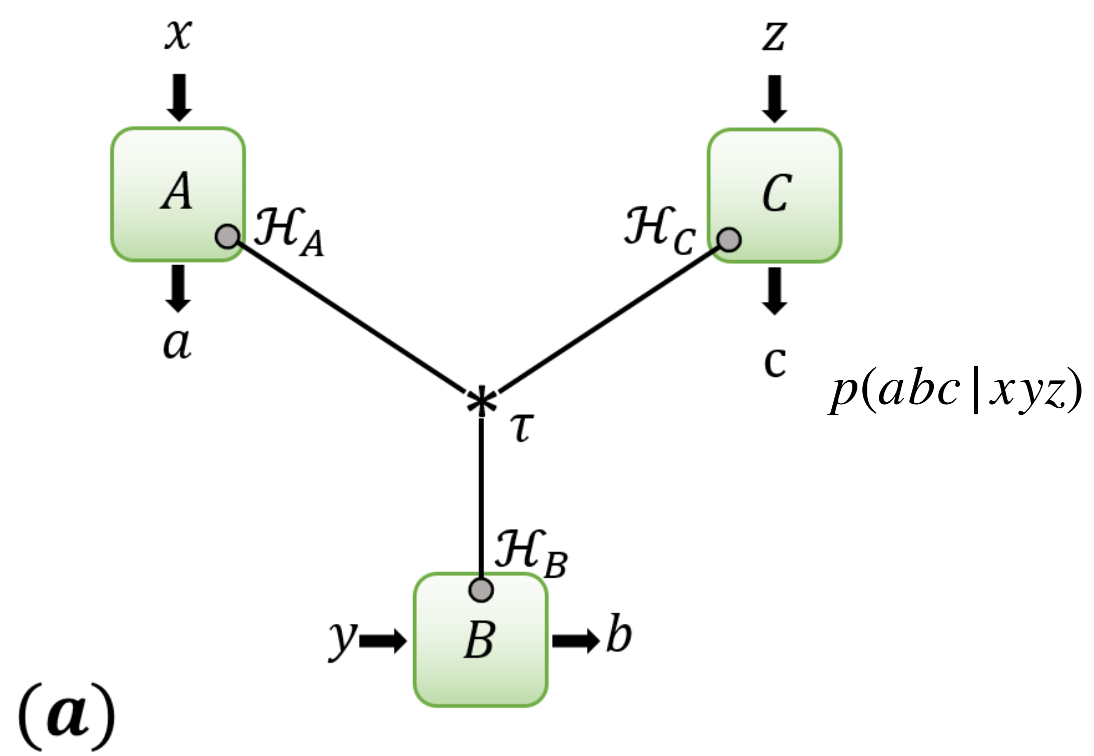

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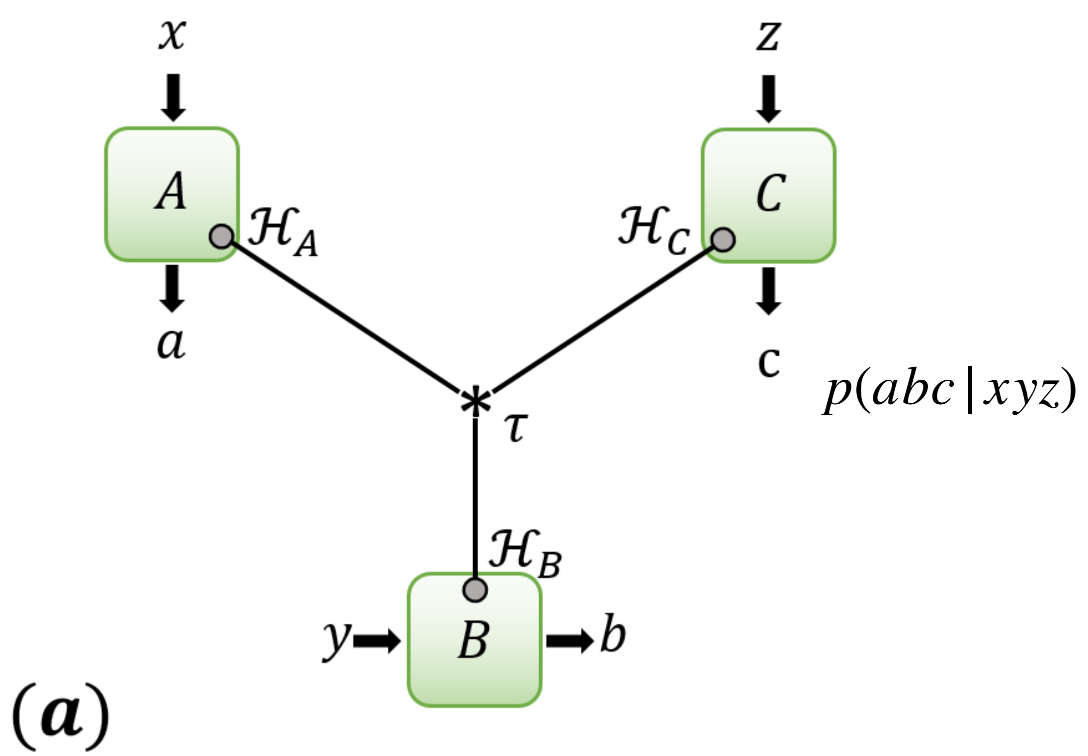
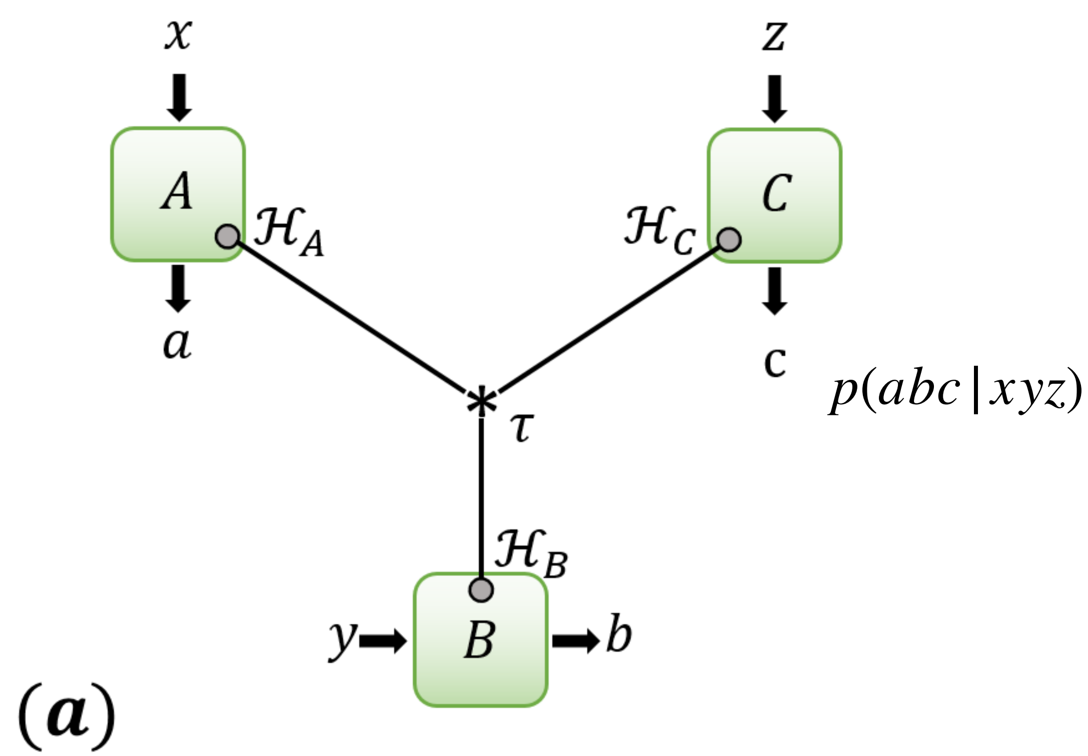


- Hilbert space $H = H_A \otimes H_B \otimes H_C$ with a shared state τ
- PVMs $\{A_{a|x}\}, \{B_{b|y}\}, \{C_{c|z}\}$
- Born's rule:
  $p(abc\,|\,xyz) = \mathrm{Tr}(\tau(A_{a|x} \otimes B_{b|y} \otimes C_{c|z})) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z})$

Tripartite Bell scenario: certification problem



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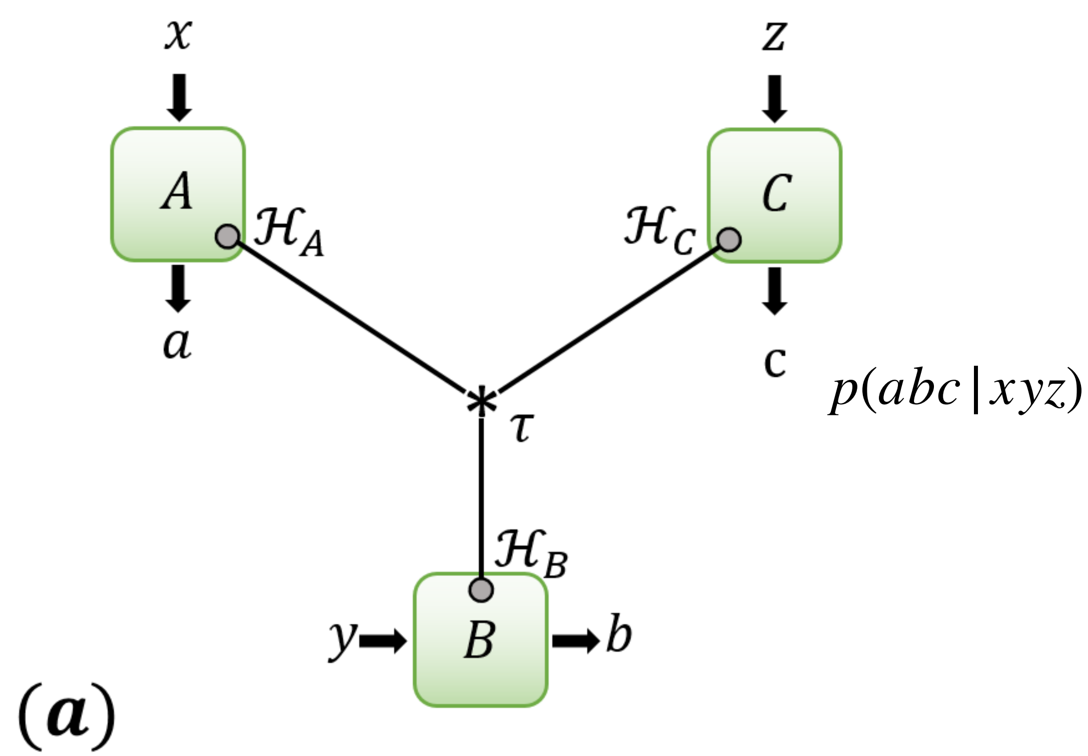

- Conversely, given $\overrightarrow{P} = \{p(abc\,|\,xyz)\}$, is it compatible with some tripartite Bell experiment?

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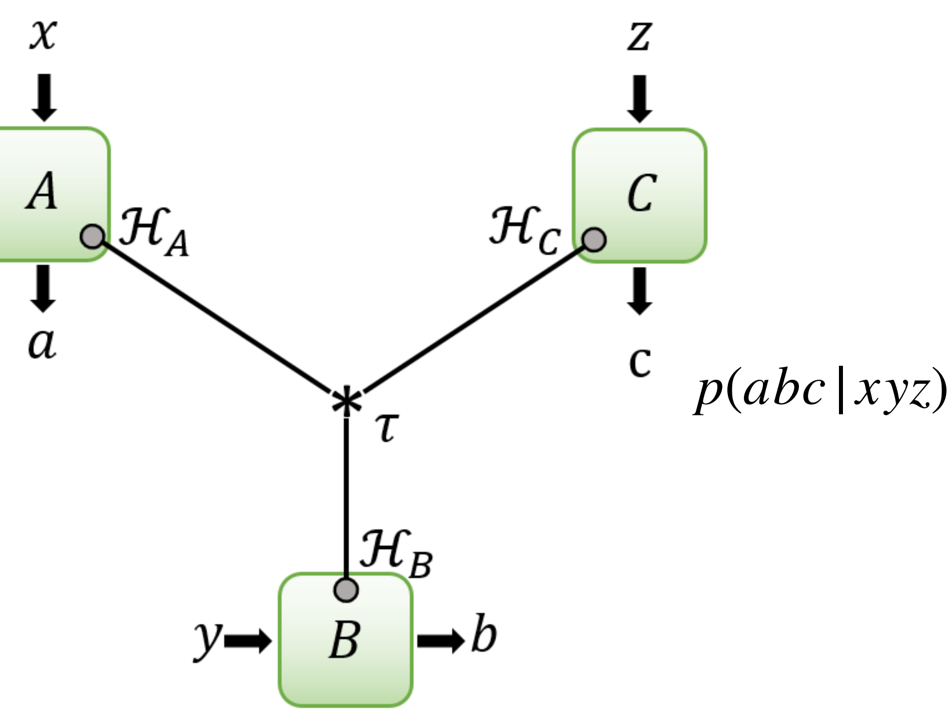

$(\boldsymbol{a})$

- Conversely, given $\overrightarrow{P} = \{p(abc\,|\,xyz)\}$, is it compatible with some tripartite Bell experiment?

- I.e does it exist some H, τ, $\{A_{a|x}\}$, … such that $p(abc\,|\,xyz) = \ldots$ Is $\overrightarrow{P} \in C_{qa}$?

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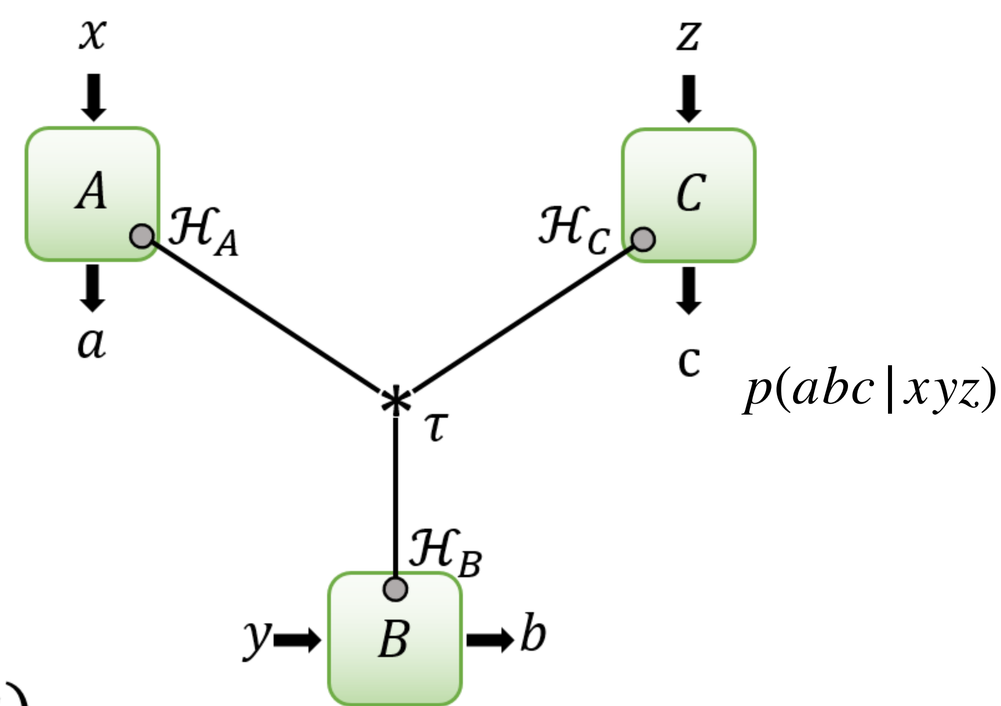

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- Useful for e.g. device-independent quantum cryptography/quantum key distribution etc.

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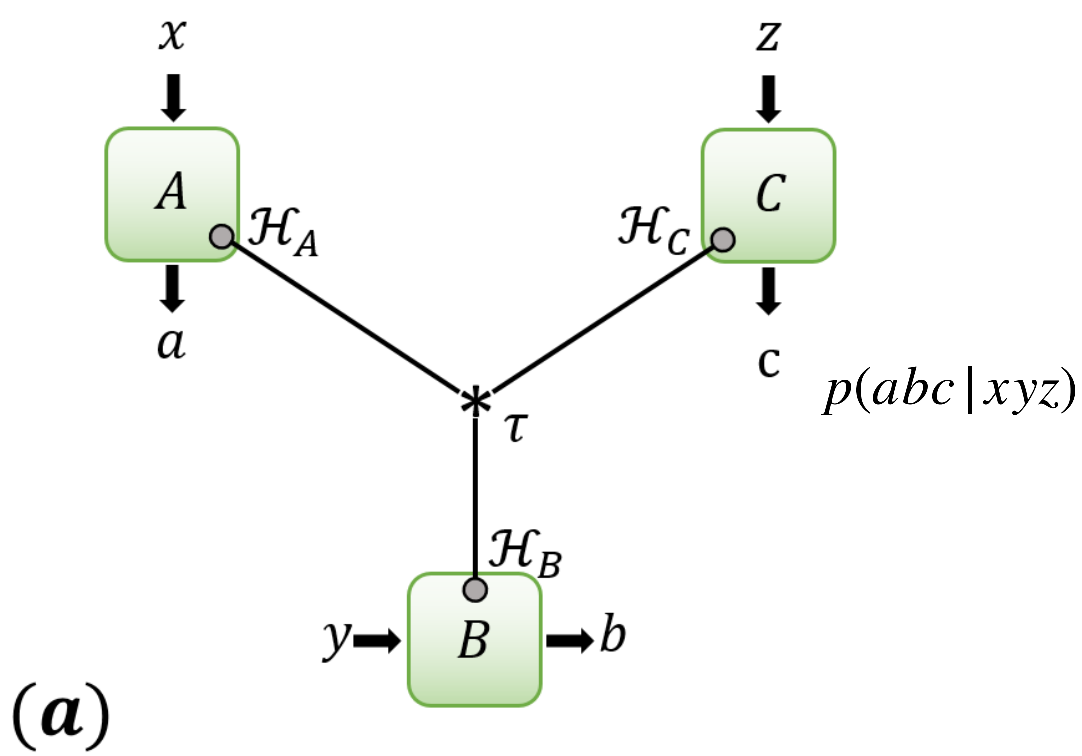

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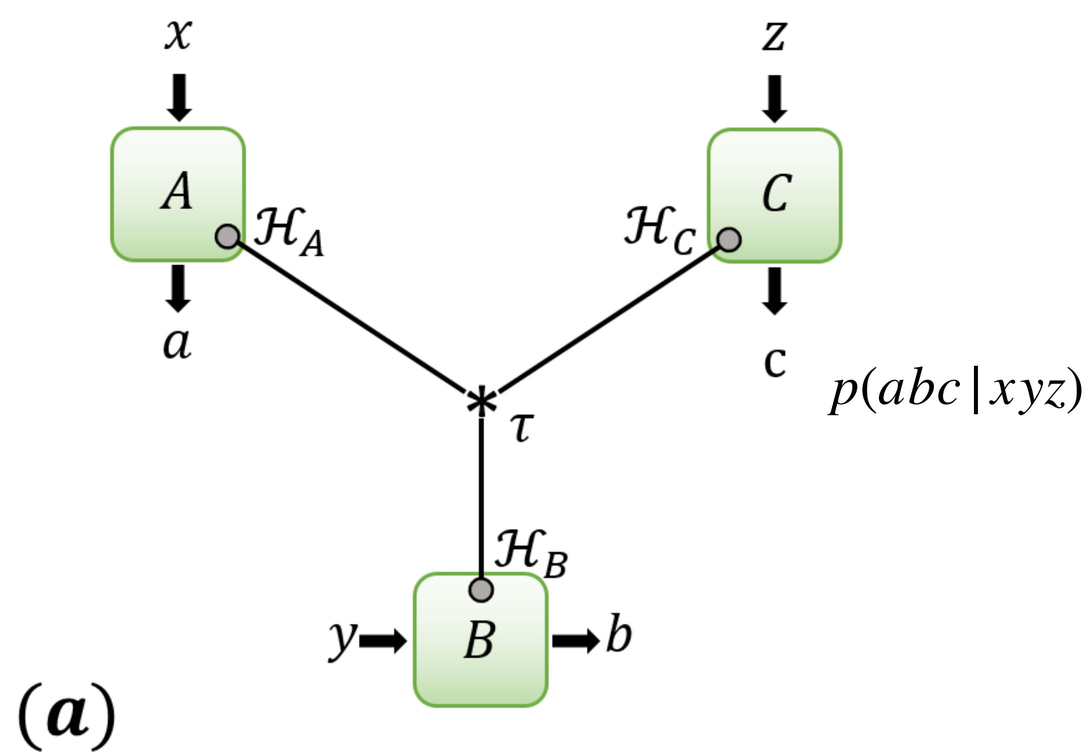

- Inner approximation: calculating all possible $\overrightarrow{P}$ over Hilbert spaces of all dimension, with e.g. gradient-descent.

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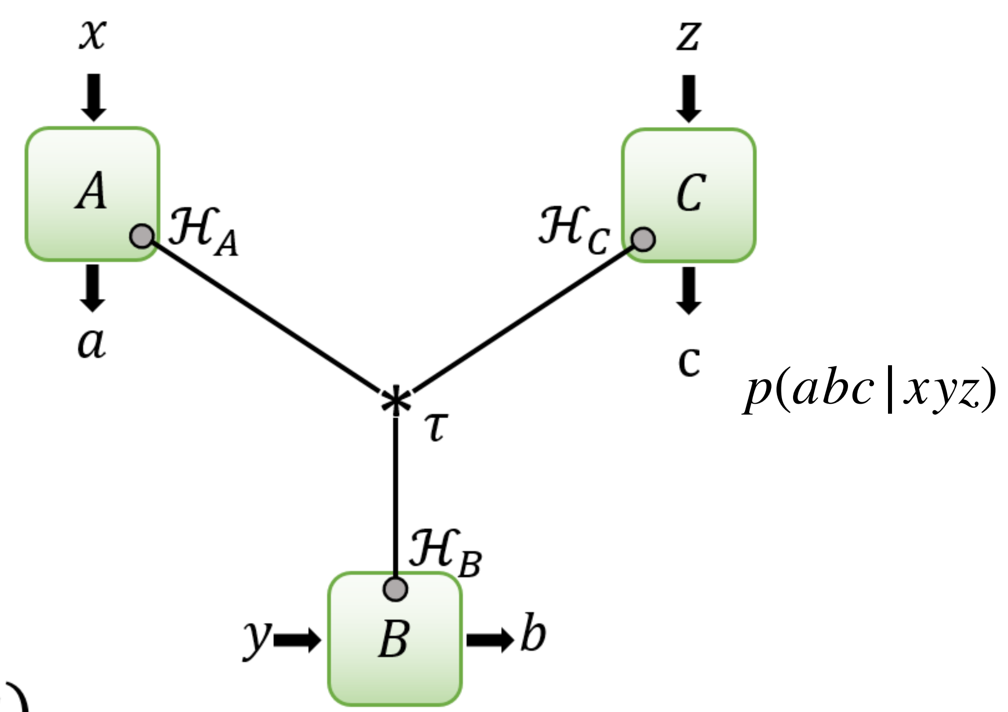

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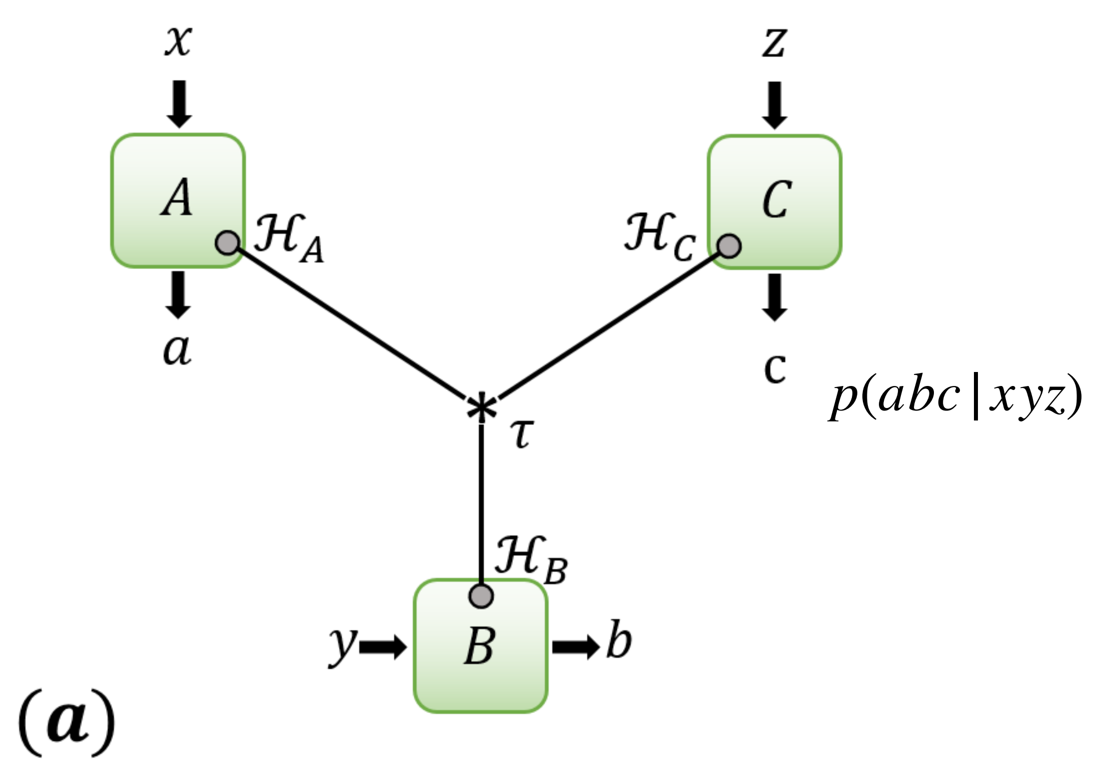

- Inner approximation: calculating all possible $\overrightarrow{P}$ over Hilbert spaces of all dimension, with e.g. gradient-descent.
- Might miss some important distributions!
- Outer approximation: NPA hierarchy [Navascués et al., 2008]

NPA hierarchy: a hierarchy of necessary conditions

- Will sketch, have condition $\Gamma^n, n \geq 2$, such that $\overrightarrow{P} \in C_{qa} \implies \cdots \implies \Gamma^4 \implies \Gamma^3 \implies \Gamma^2$.

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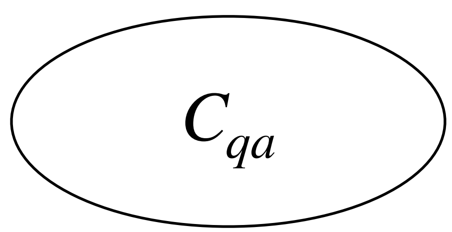

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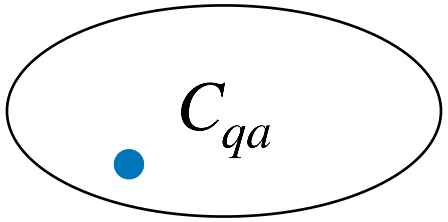

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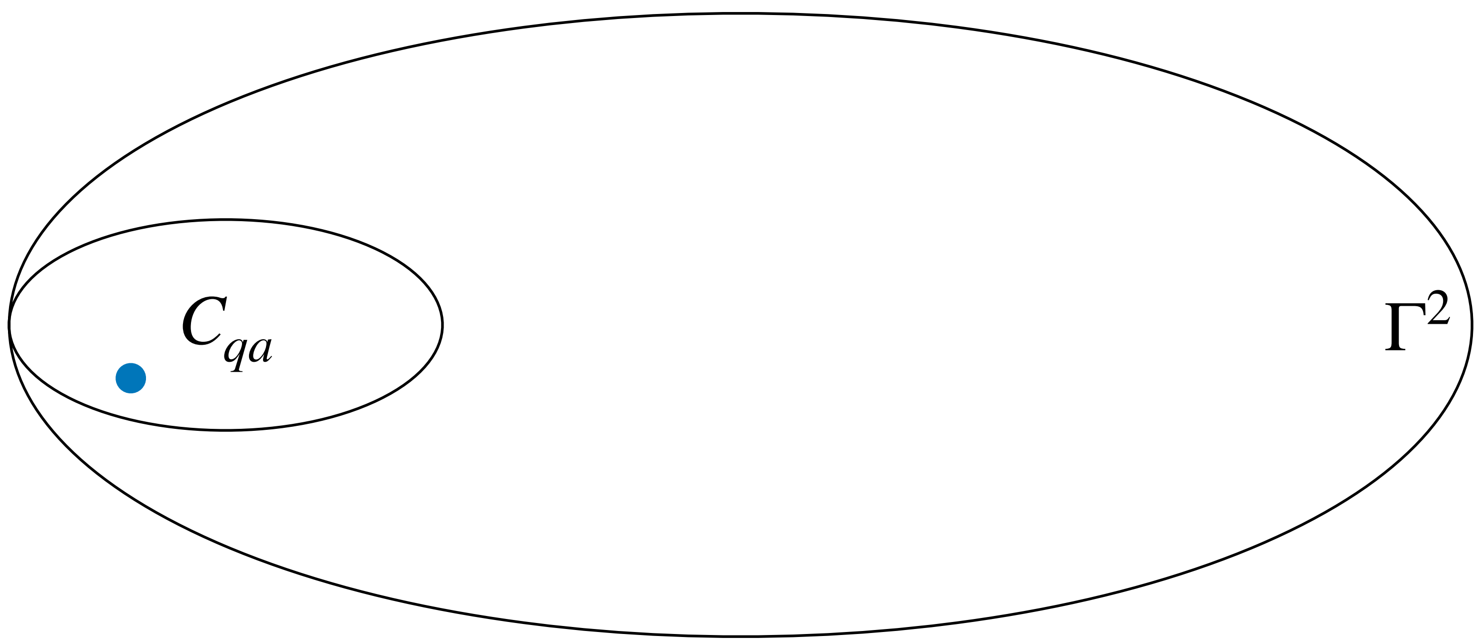

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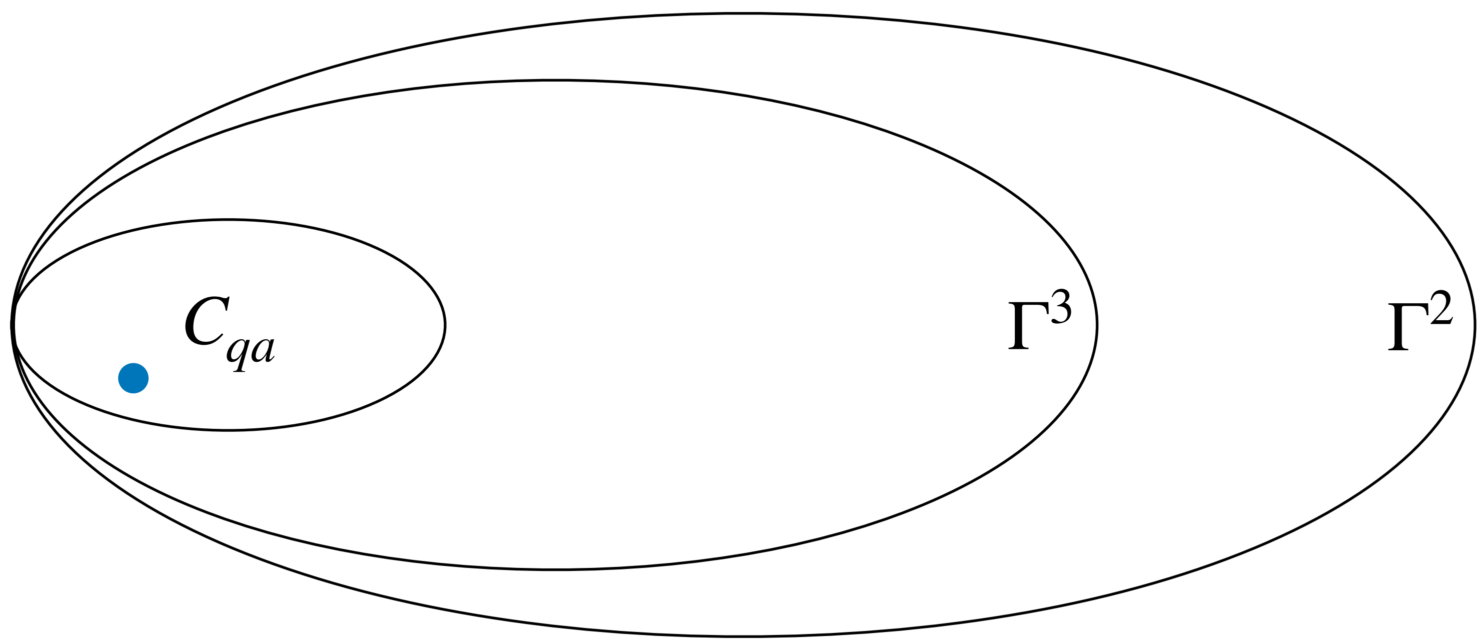

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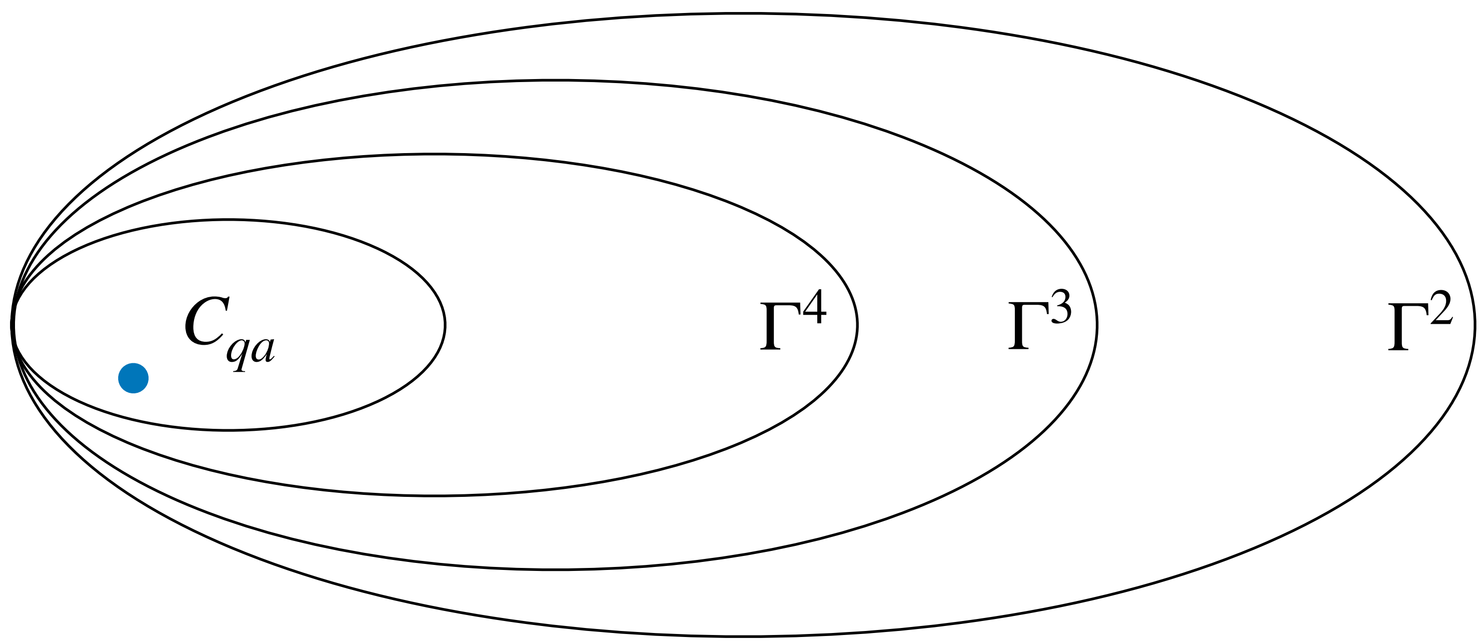

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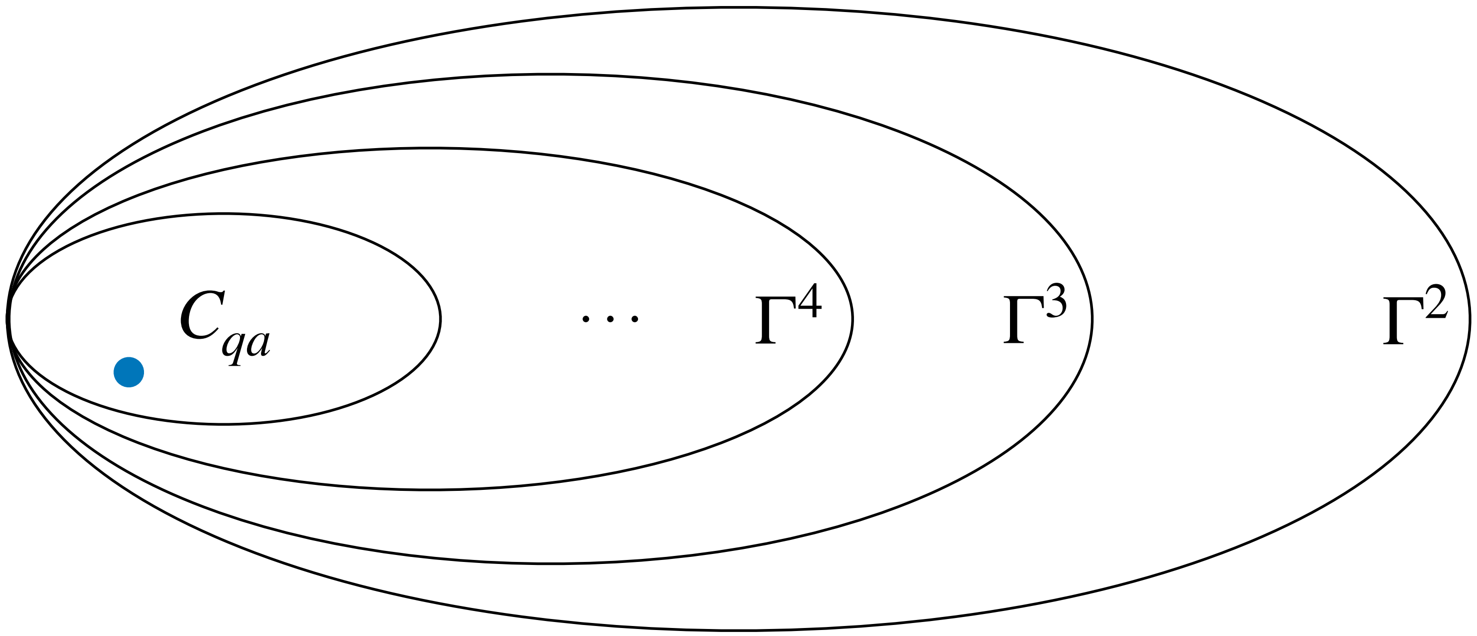

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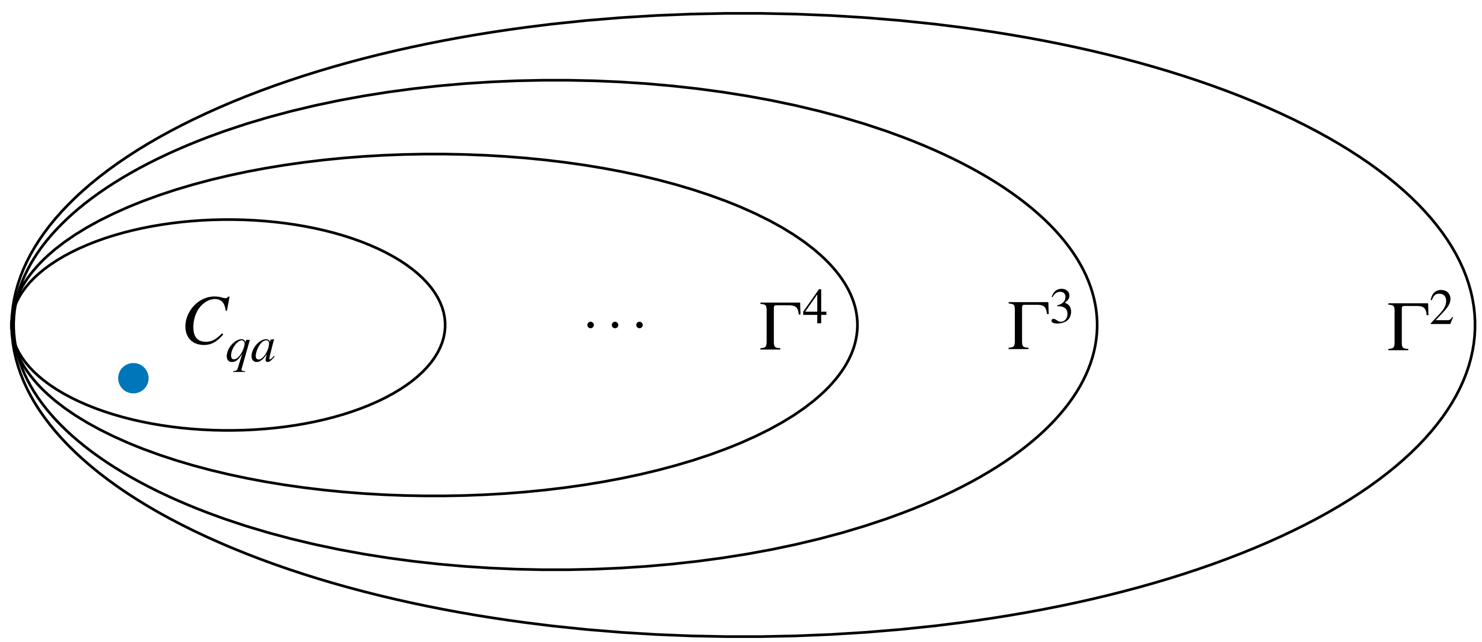

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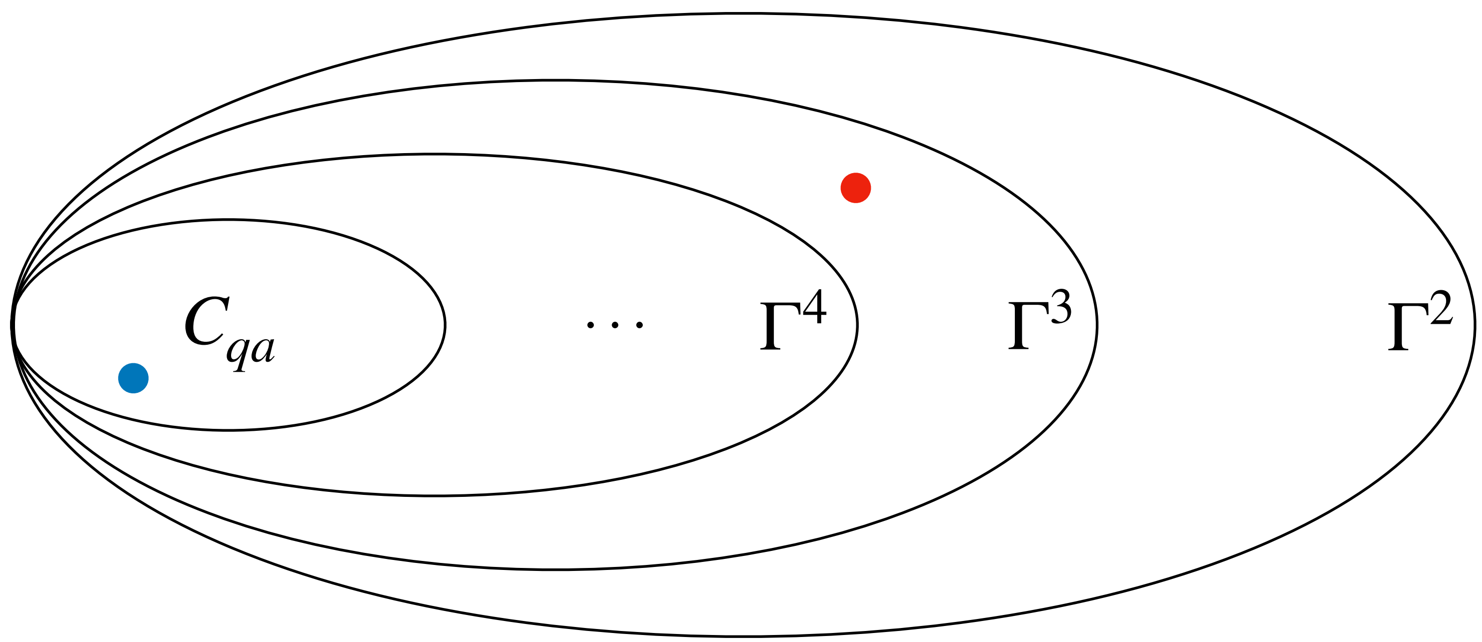

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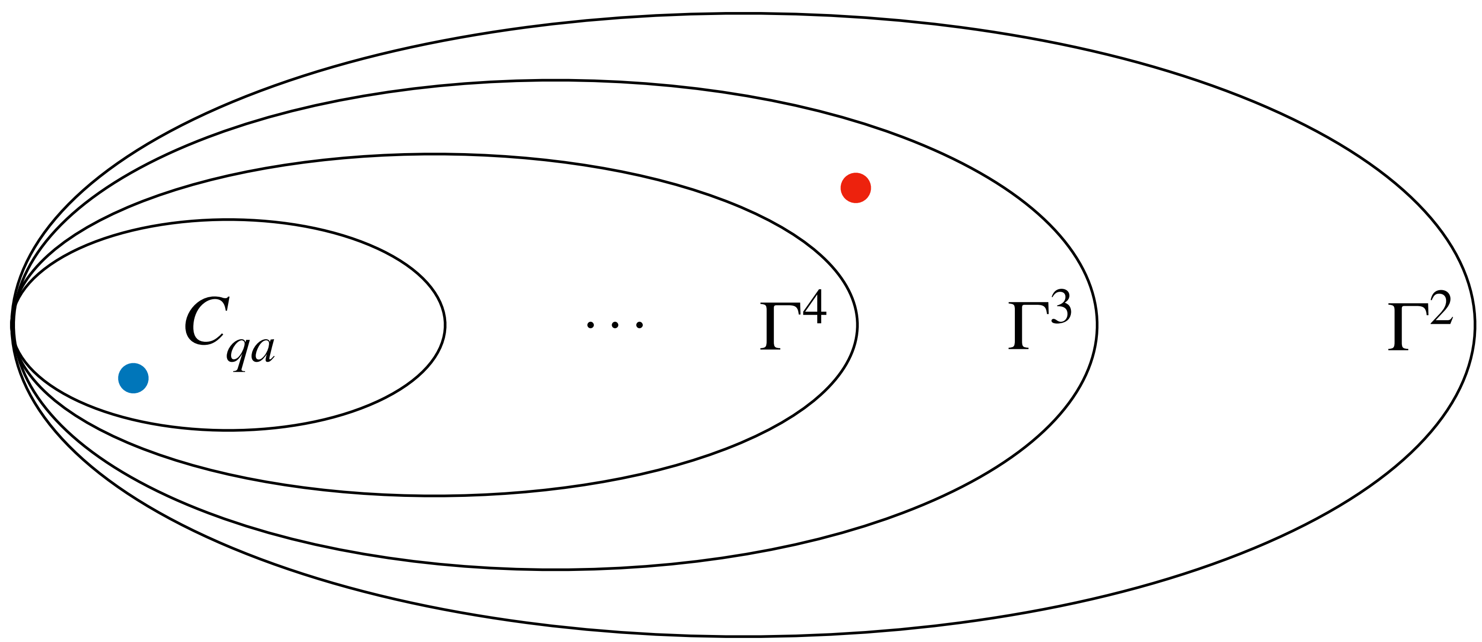

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- Testing C_{qa} from the outside.

NPA hierarchy: moment matrix Γ^2

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- Suppose state&PVMs s.t. $p(abc \mid xyz) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z})$, easy to calculate $\mathrm{Tr}_\tau(A_{a|x})$, $\mathrm{Tr}_\tau(A^\dagger_{a|x}C_{c|z})$, …

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$$
\begin{array}{c c}
 & \begin{array}{ccccc} \mathbb{1} & A_{a|x} & B_{b|y} & C_{c|z} & \cdots \end{array} \\
\begin{array}{c} \mathbb{1} \\ (A_{a|x})^\dagger \\ (B_{b|y})^\dagger \\ (C_{c|z})^\dagger \\ \cdots \end{array} &
\begin{bmatrix}
1 & \mathrm{Tr}_\tau\left(A_{a|x}\right) & \mathrm{Tr}_\tau\left(B_{b|y}\right) & \mathrm{Tr}_\tau\left(C_{c|z}\right) & \\
 & \mathrm{Tr}_\tau\left(A_{a|x}\right) & \mathrm{Tr}_\tau\left(A_{a|x}B_{b|y}\right) & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & \\
 & & \mathrm{Tr}_\tau\left(B_{b|y}\right) & \mathrm{Tr}_\tau\left(B_{b|y}C_{c|z}\right) & \\
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 & & & &
\end{bmatrix}
\end{array}
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 & & \mathrm{Tr}_\tau\left(B_{b|y}\right) & \mathrm{Tr}_\tau\left(B_{b|y}C_{c|z}\right) & \\
 & & & \mathrm{Tr}_\tau\left(C_{c|z}\right) & \\
 & & & &
\end{bmatrix}
\end{array}
$$

- Rule:
$$\Gamma_{B_{b|y},B_{b|y}} = \mathrm{Tr}_\tau(B^\dagger_{b|y}B_{b|y}) = \mathrm{Tr}_\tau(B_{b|y}) = \mathrm{Tr}_\tau(\mathrm{Id}^\dagger \cdot B_{b|y}) = \Gamma_{1,B_{b|y}}$$

NPA hierarchy: moment matrix Γ^2

- Zoom out to length 2

$$\Gamma_{A_{a|x}B_{b|y},C_{c|z}} = \mathrm{Tr}_\tau((A_{a|x}B_{b|y})^\dagger C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z}) = p(abc\,|\,xyz)$$

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$$
\begin{array}{c c}
 & \begin{array}{cccccc} \mathbb{1} & A_{a|x} & B_{b|y} & C_{c|z} & A_{a|x}A_{a'|x'} & \cdots \end{array} \\
\begin{array}{c} \mathbb{1} \\ (A_{a|x})^\dagger \\ (B_{b|y})^\dagger \\ (C_{c|z})^\dagger \\ (A_{a|x}A_{a'|x'})^\dagger \\ (A_{a|x}B_{b|y})^\dagger \\ \cdots \end{array} &
\begin{bmatrix} 1 & & & & & \\ & & & & & \\ & & \mathrm{Tr}_\tau\left(B_{b|y}\right) & & & \\ & & & & & \\ & & & \mathrm{Tr}_\tau\left(A_{a'|x'}A_{a|x}C_{c|z}\right) & & \\ & & & p(abc|xyz) & & \\ & & & & & \end{bmatrix}
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\end{array}$$

- Γ^2 is semidefinite positive, symmetric, satisfies many linear constraints…

NPA hierarchy: moment matrix Γ^3 and more

- Longer indices, such as $A_{a|x}B_{b|y}C_{c|z}$, of length 3, to get a bigger matrix Γ^3.

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1 & & & & \\
 & & & & \\
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- Containing Γ^2 as a submatrix: $\Gamma^3 \implies \Gamma^2$.

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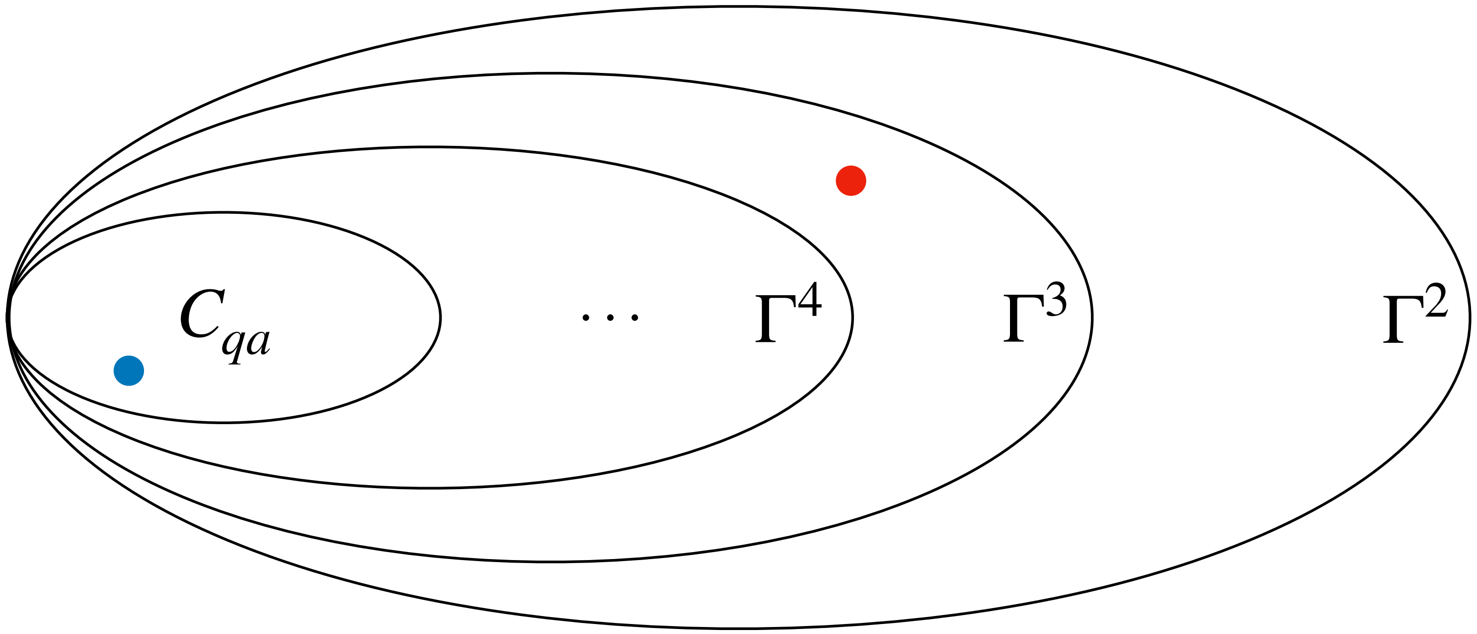
 & & & &
\end{array}\right]
\end{array}$$

- Containing Γ^2 as a submatrix: $\Gamma^3 \implies \Gamma^2$.

- Repeat to get $\Gamma^2, \Gamma^3, \Gamma^4\ldots$ A hierarchy of moment matrices.

NPA hierarchy is necessary



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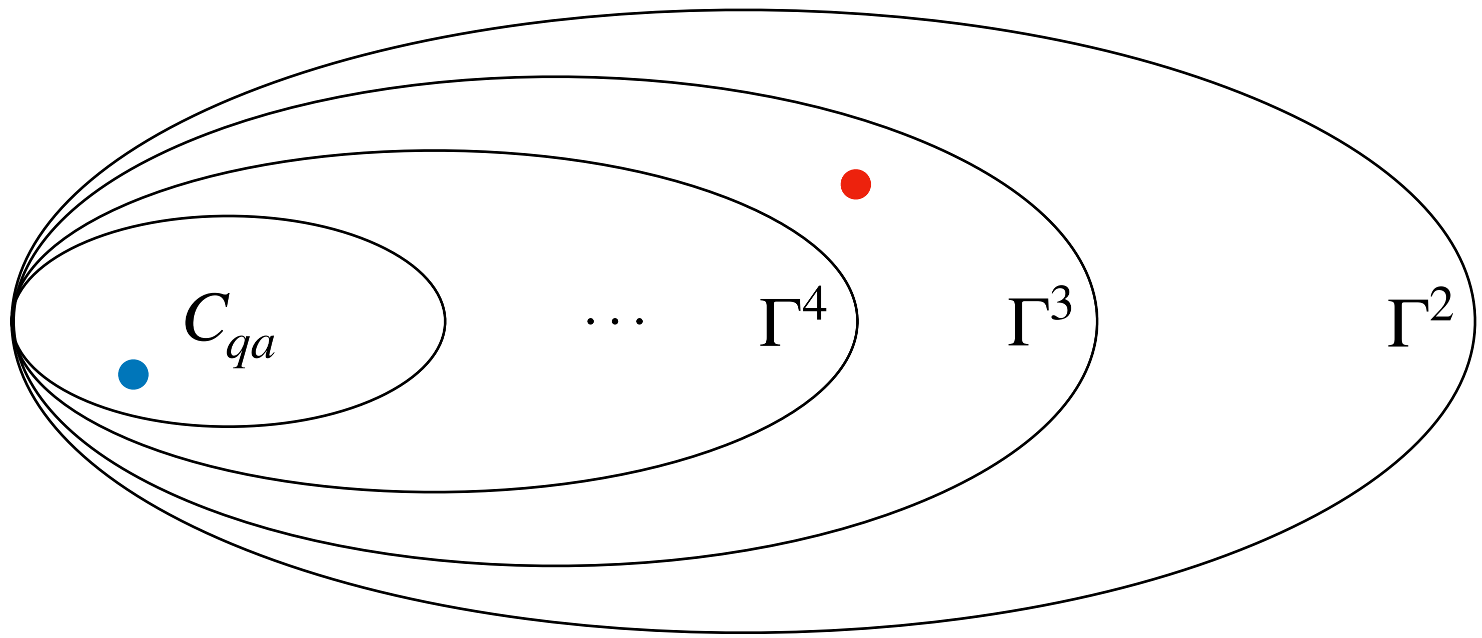

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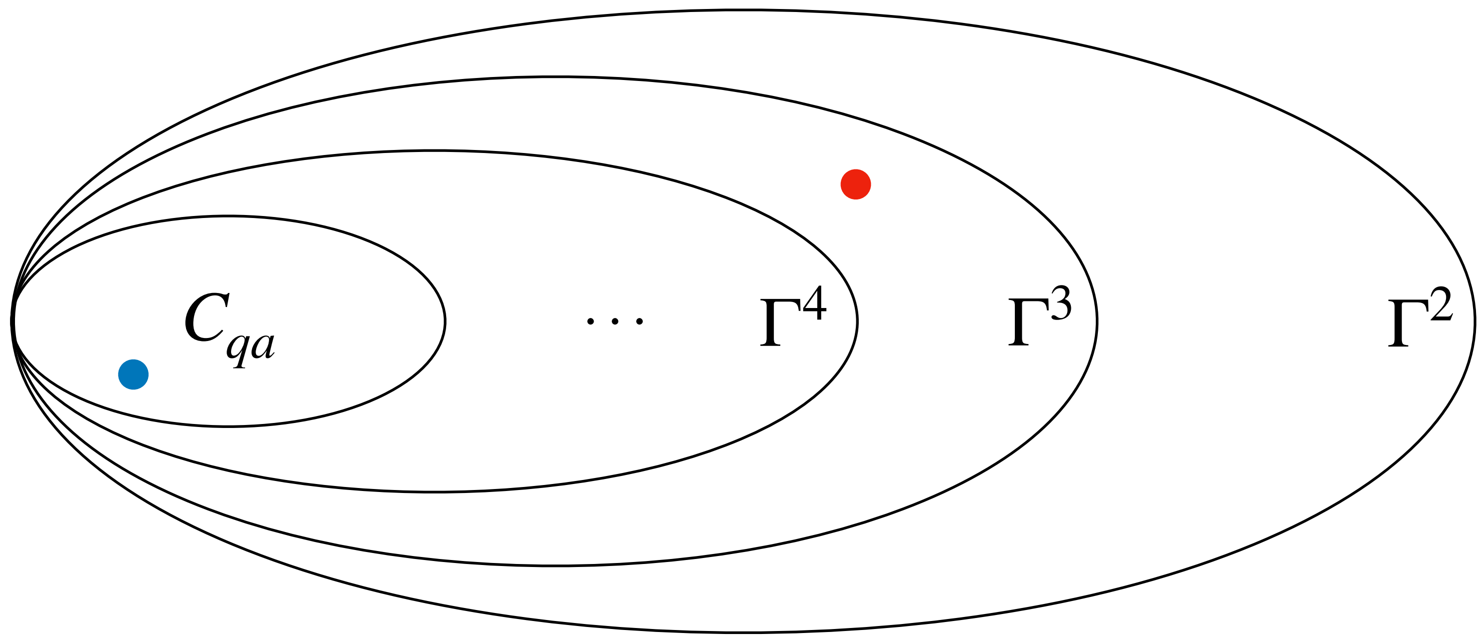

- If $\overrightarrow{P} \in C_{qa}$, then for every n there exists compatible moment matrix Γ^n.
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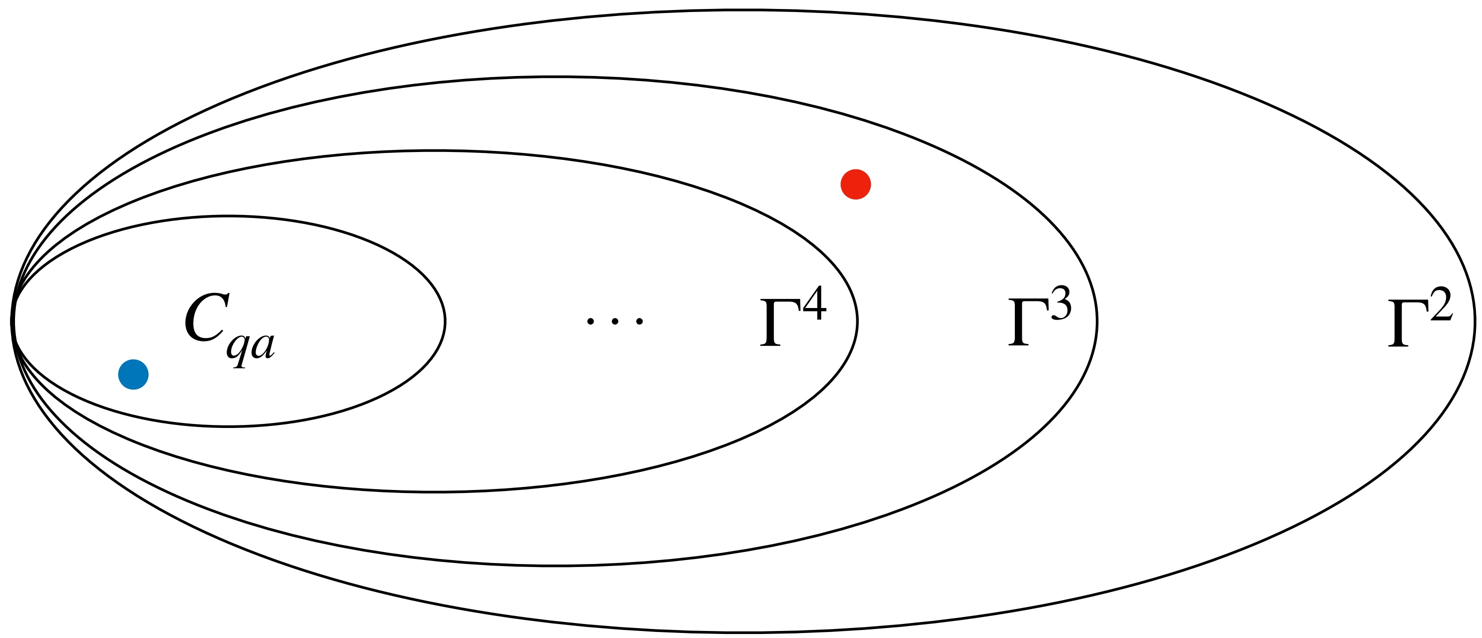

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- Semidefinite program (SDP): checking if Γ^n exists can be done with computers!

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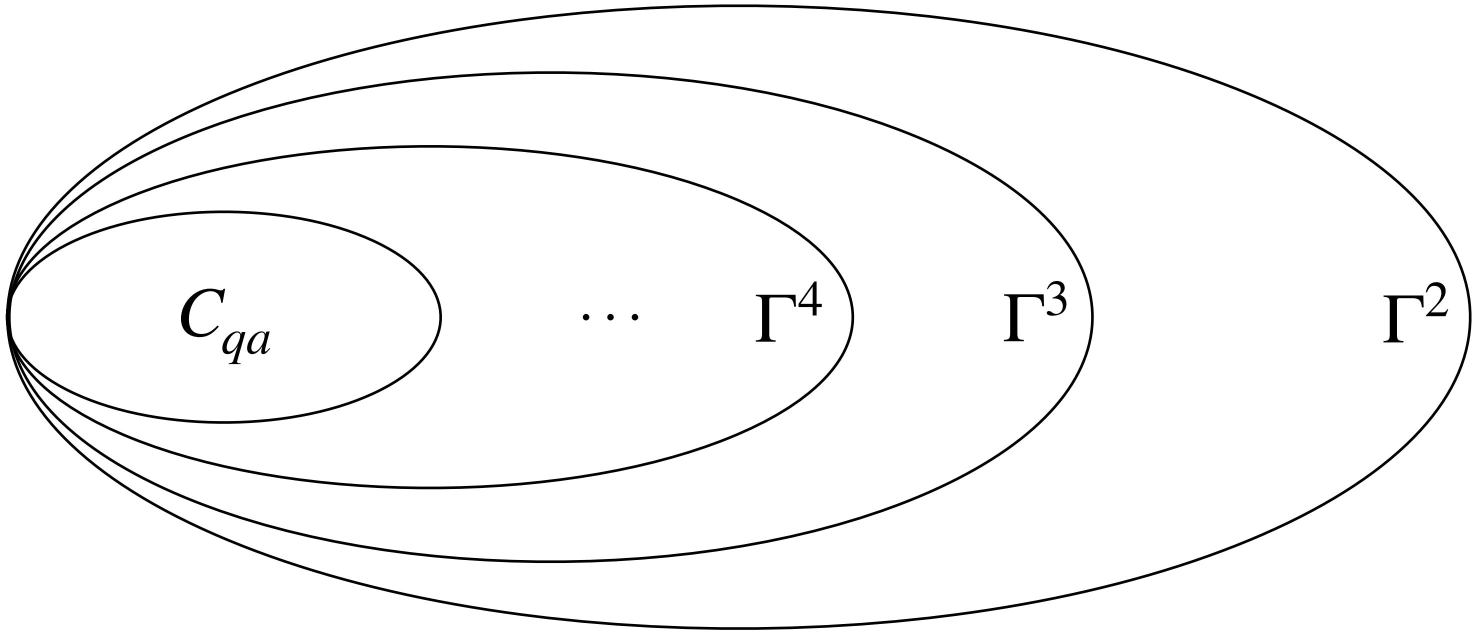

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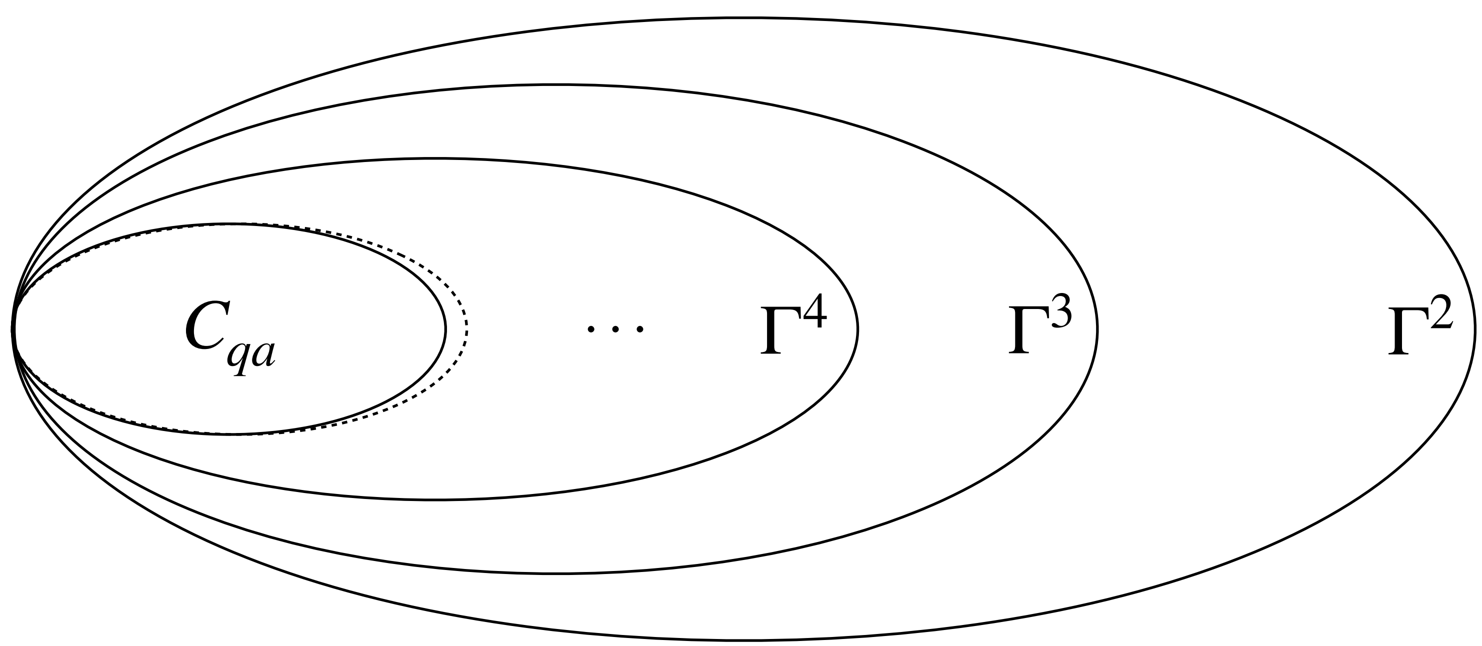
- What if $\overrightarrow{P}$ admits all Γ^n? I.e. what is the limit $lim_{n\to\infty}\Gamma^n$?



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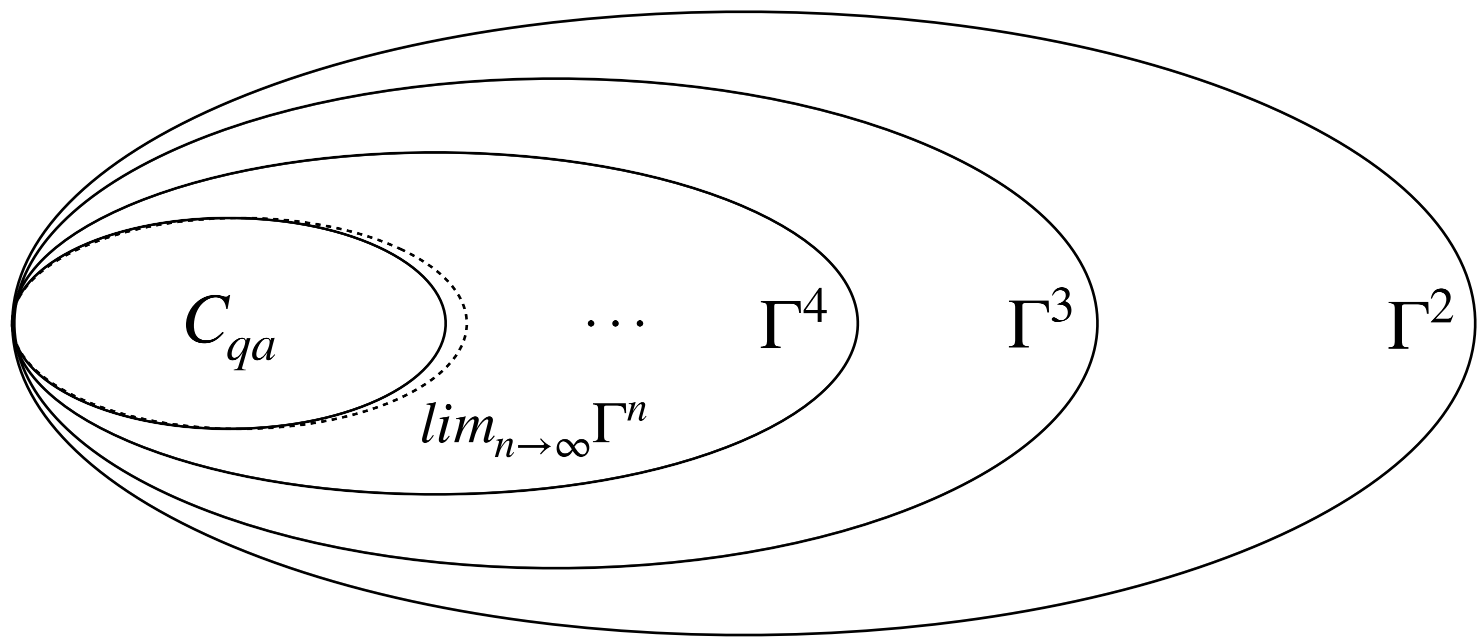

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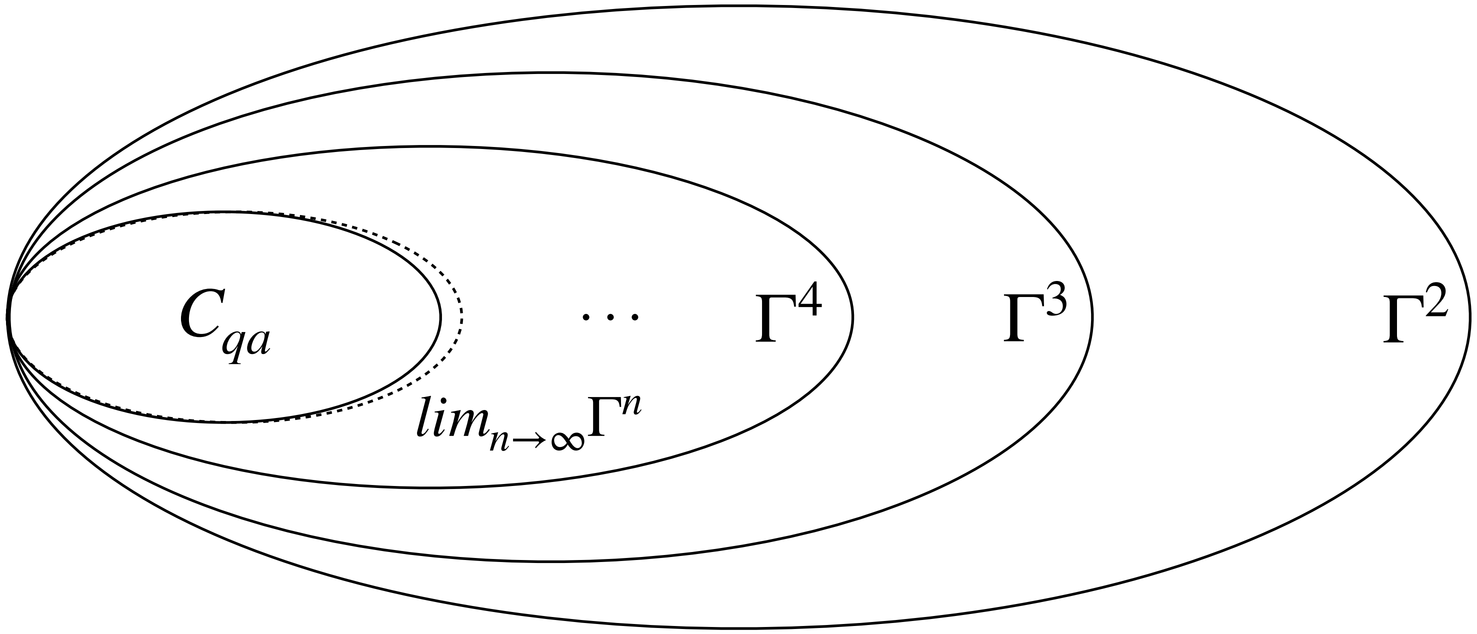

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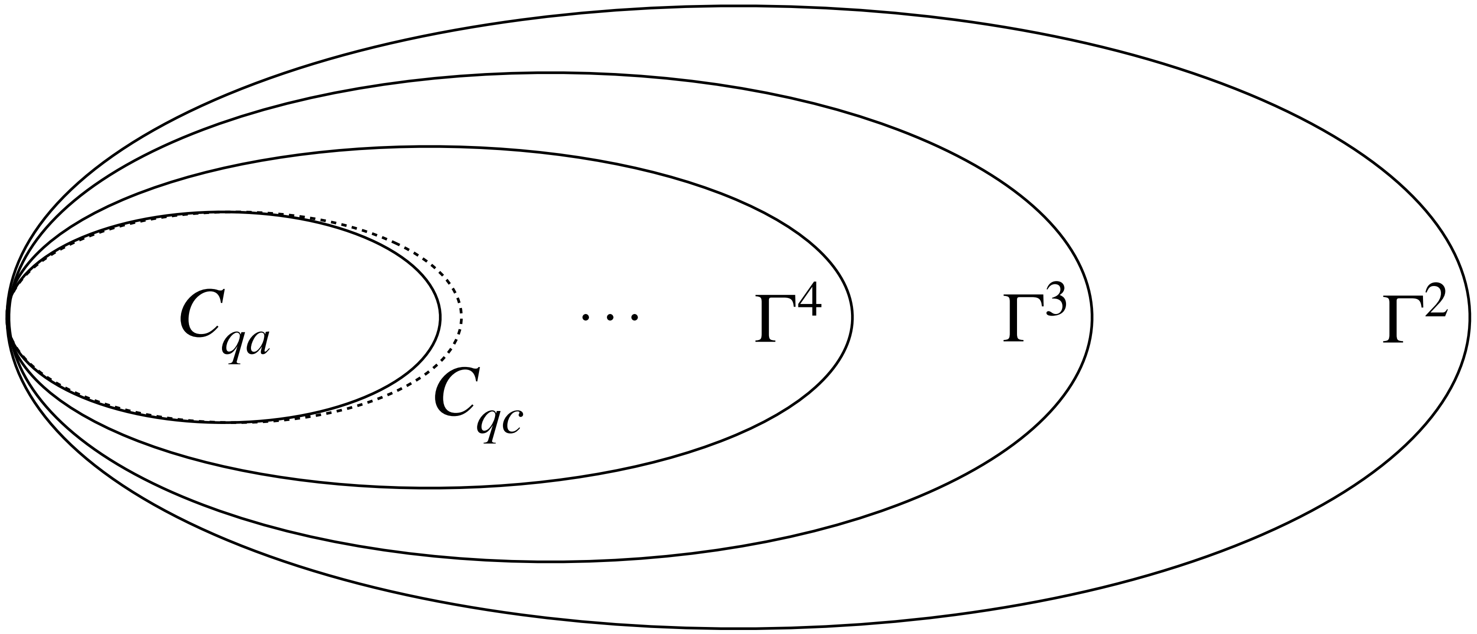
- We know $C_{qa} \subsetneq C_{qc}$ [Ji et al., 2021], but they do agree in finite dimension [Fritz, 2012].

NPA hierarchy: summary



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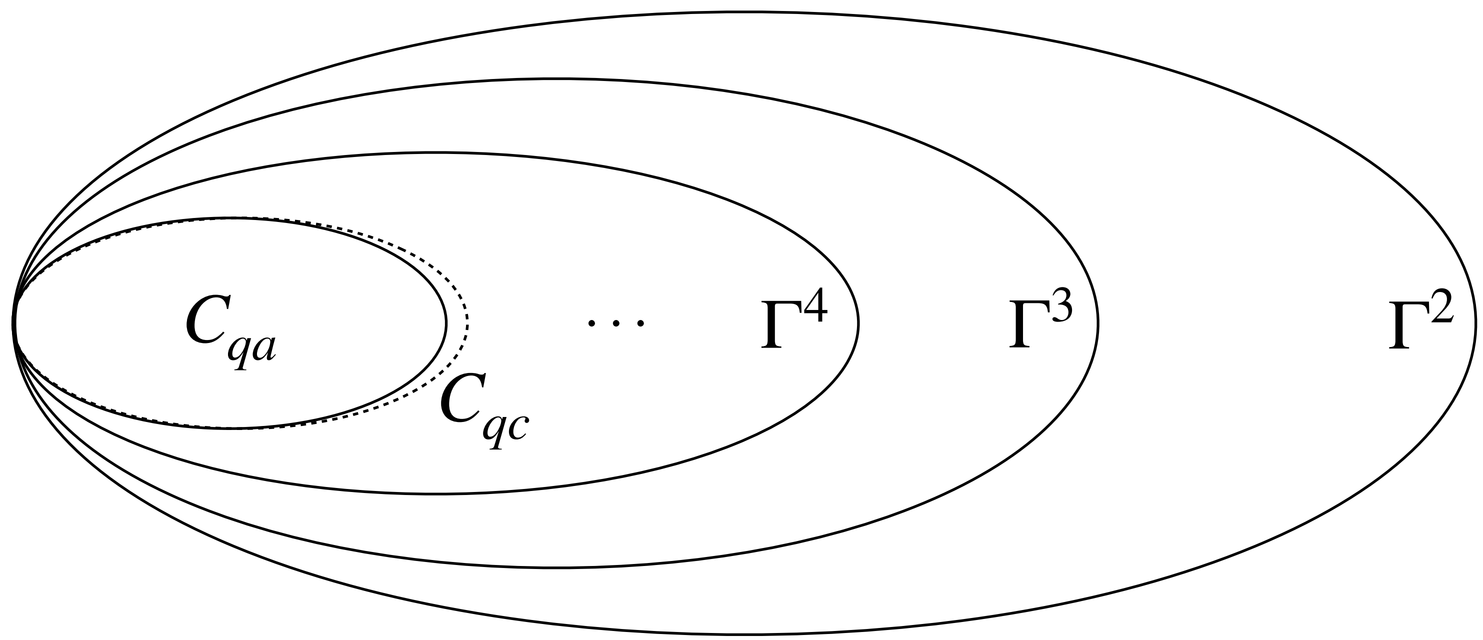

- A hierarchy Γ^n converges to commutator quantum model C_{qc} from the outside.

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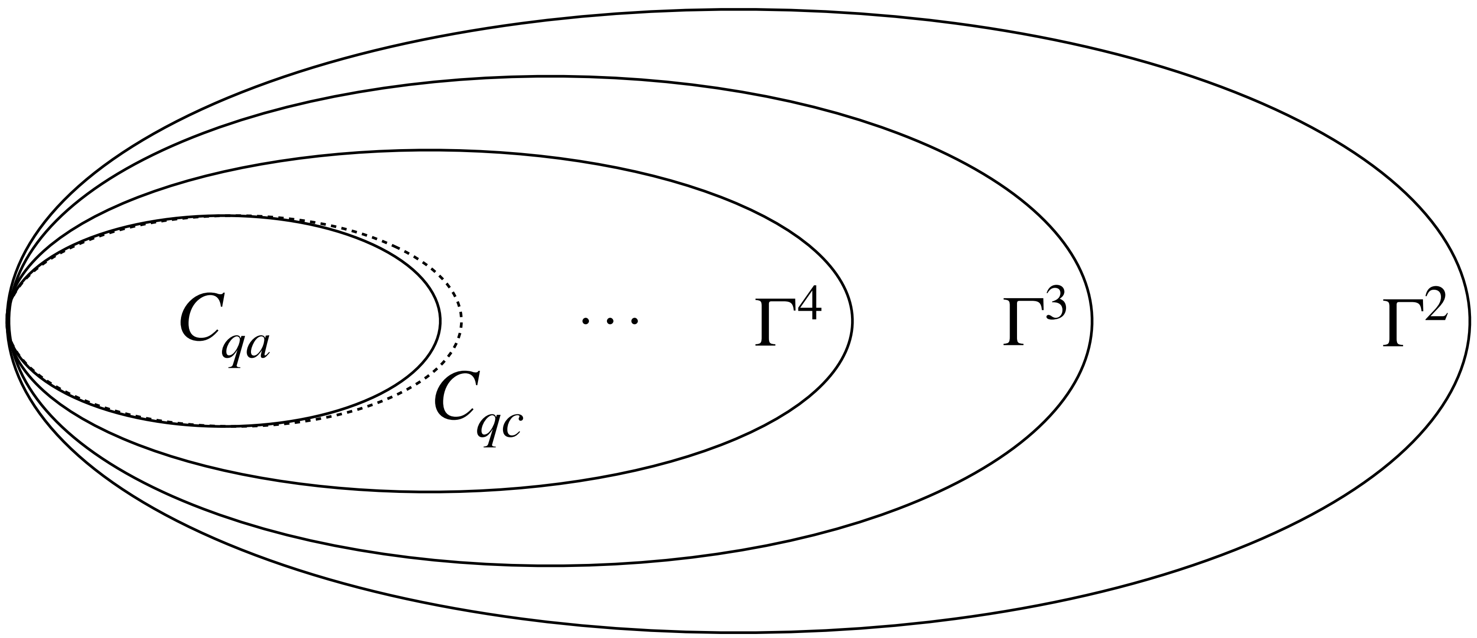

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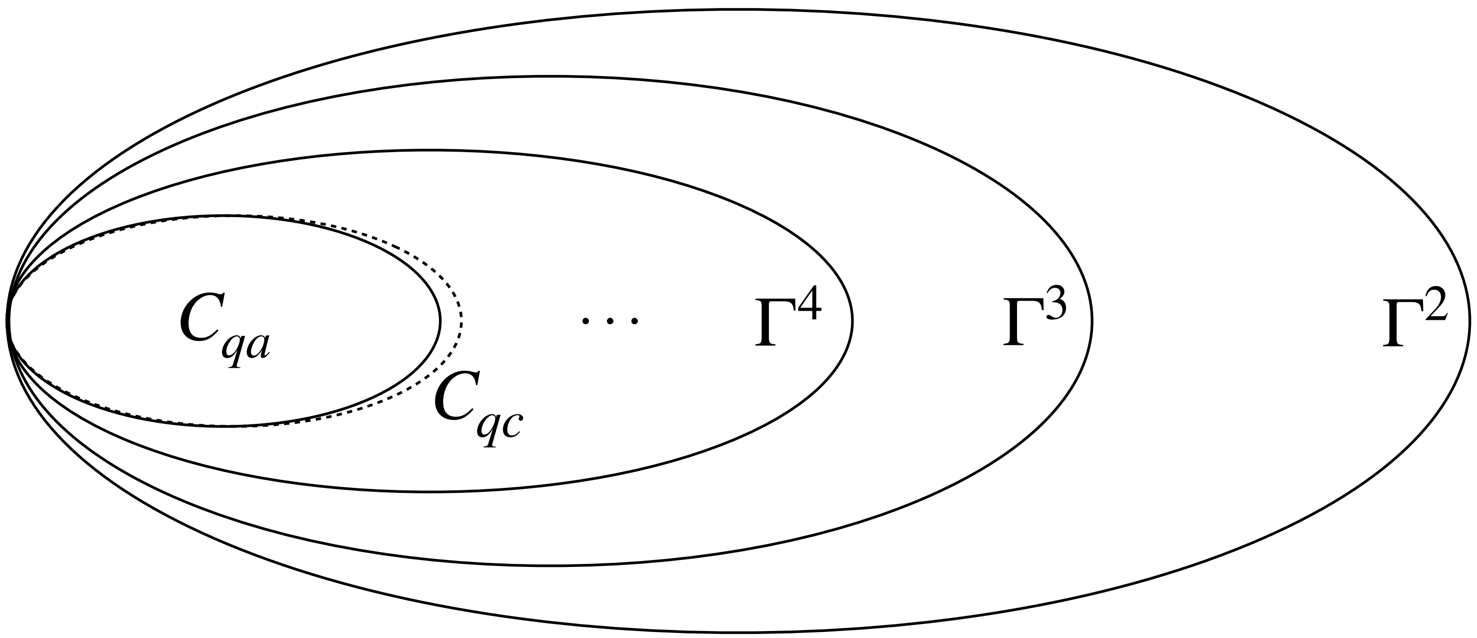
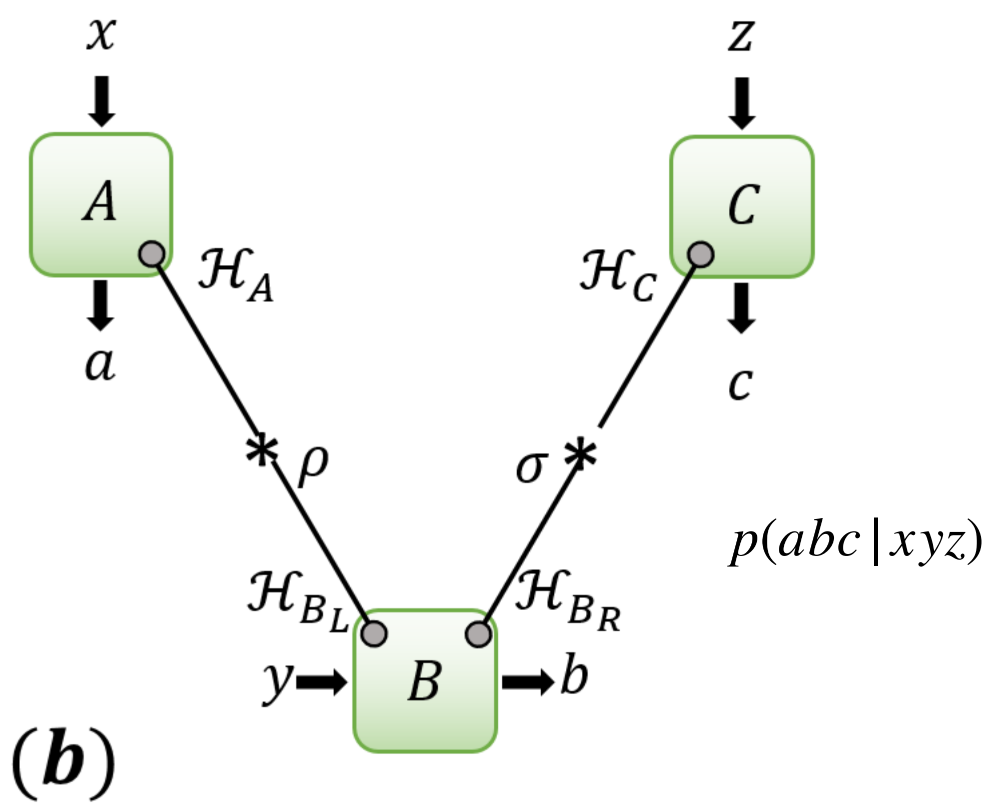


- A hierarchy Γ^n converges to commutator quantum model C_{qc} from the outside.
- In finite dimension, it converges to the usual quantum model with tensor product C_{qa}.
- Each step can be solved by computers via SDP.

Quantum bilocal scenario



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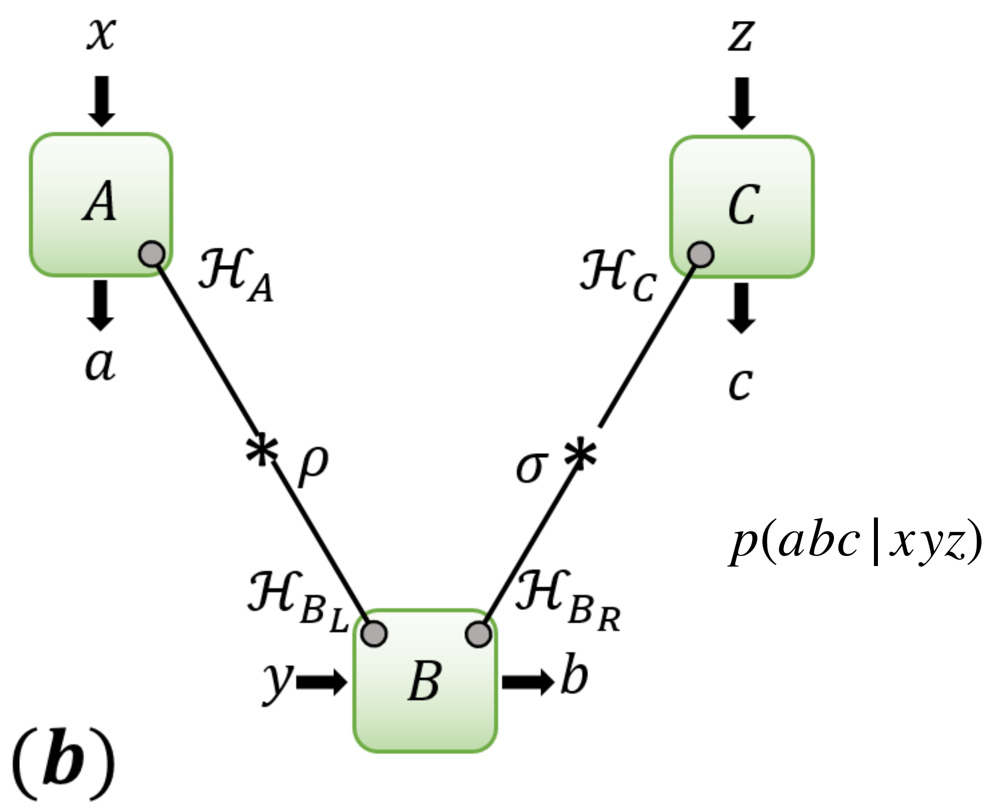

- The simplest network scenario beyond the Bell scenario.

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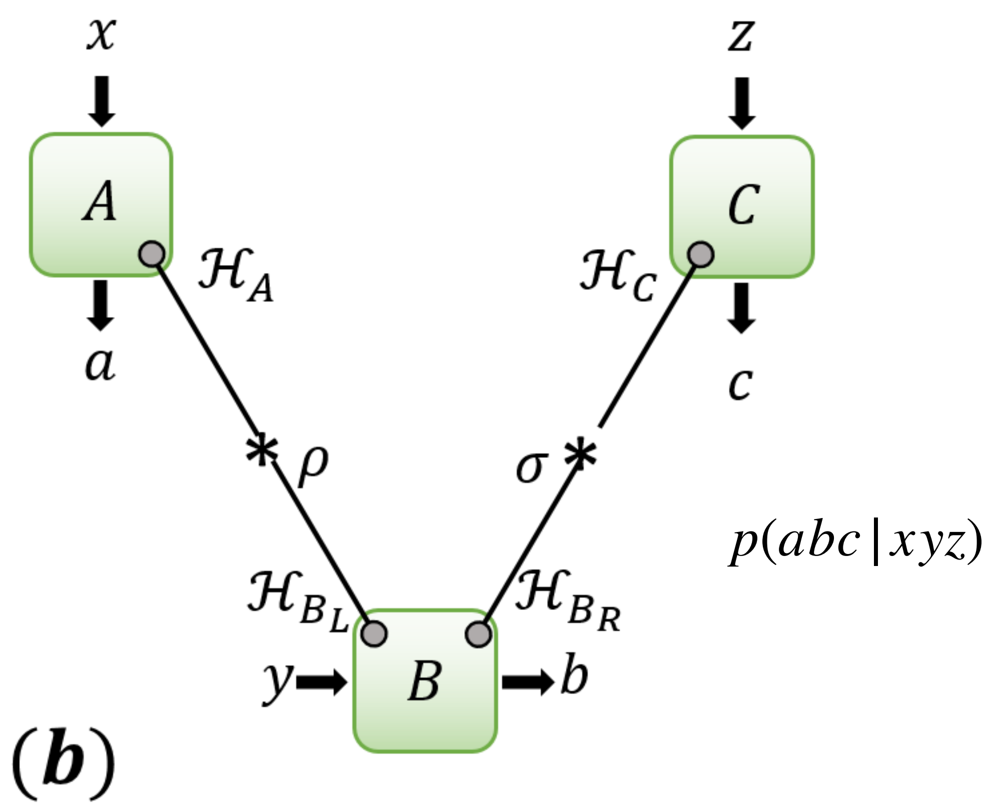
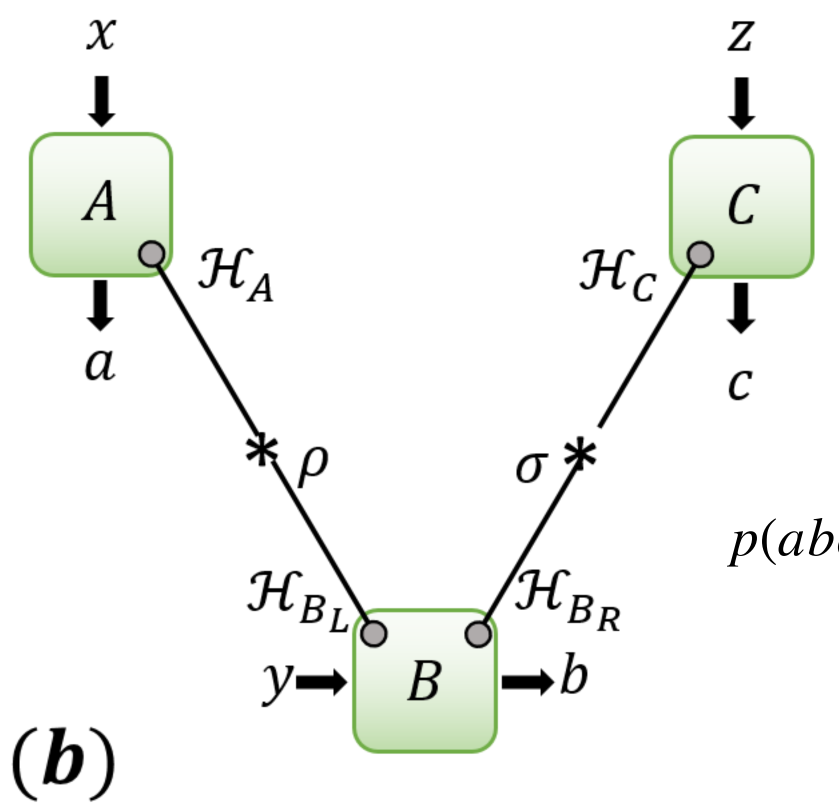


- The simplest network scenario beyond the Bell scenario.
- Entanglement swapping [Branciard et al., 2012], real quantum theory can be falsified experimentally [Renou et al., 2021], etc.

Bilocal scenario: standard QM $Q_{bilocal}$



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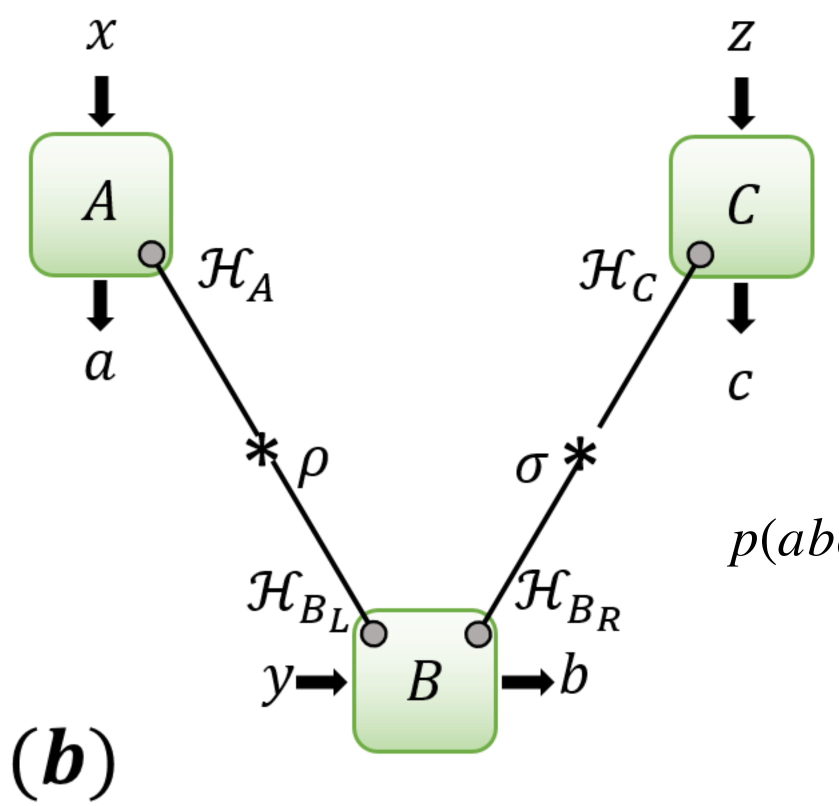

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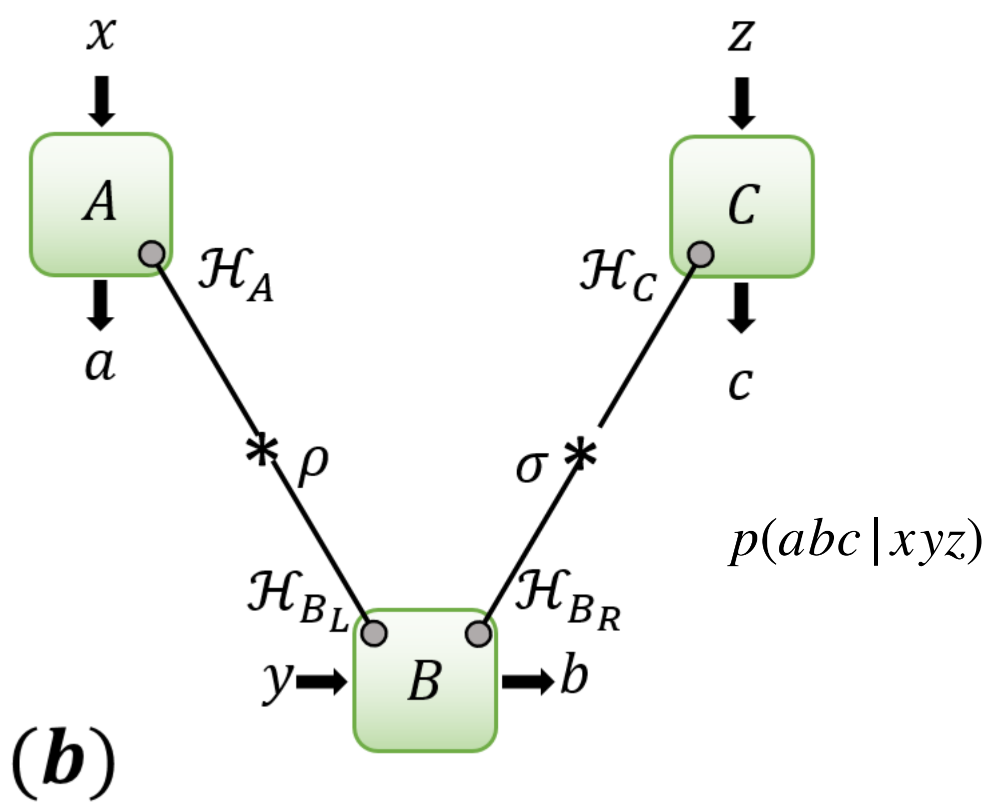

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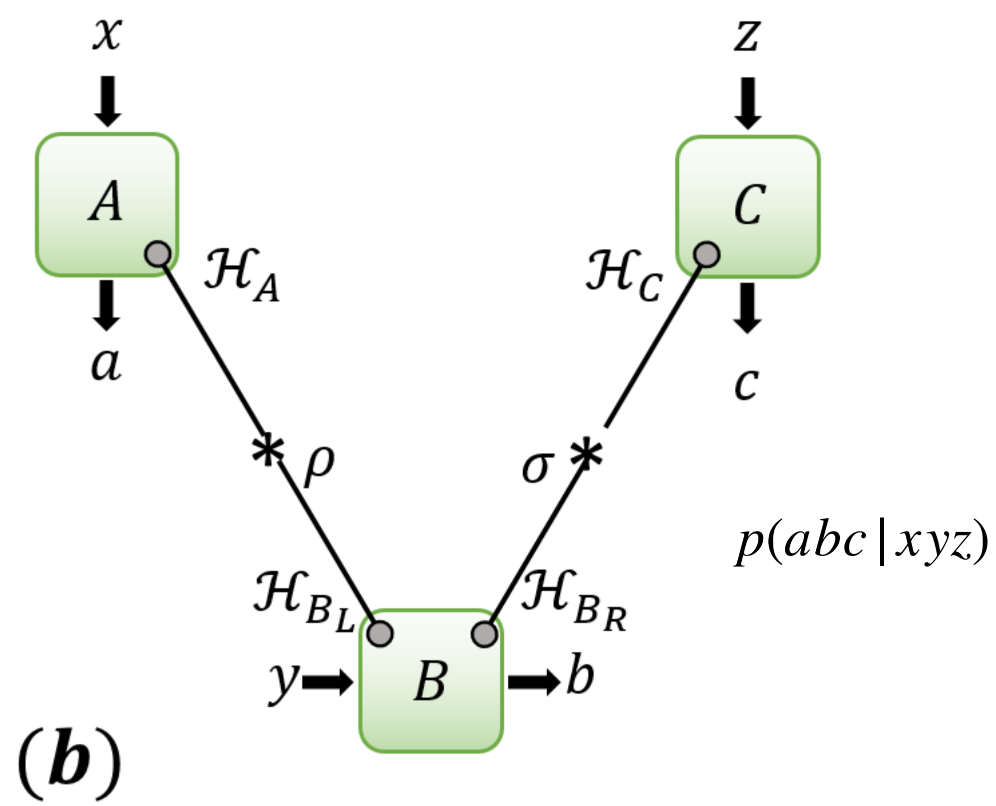

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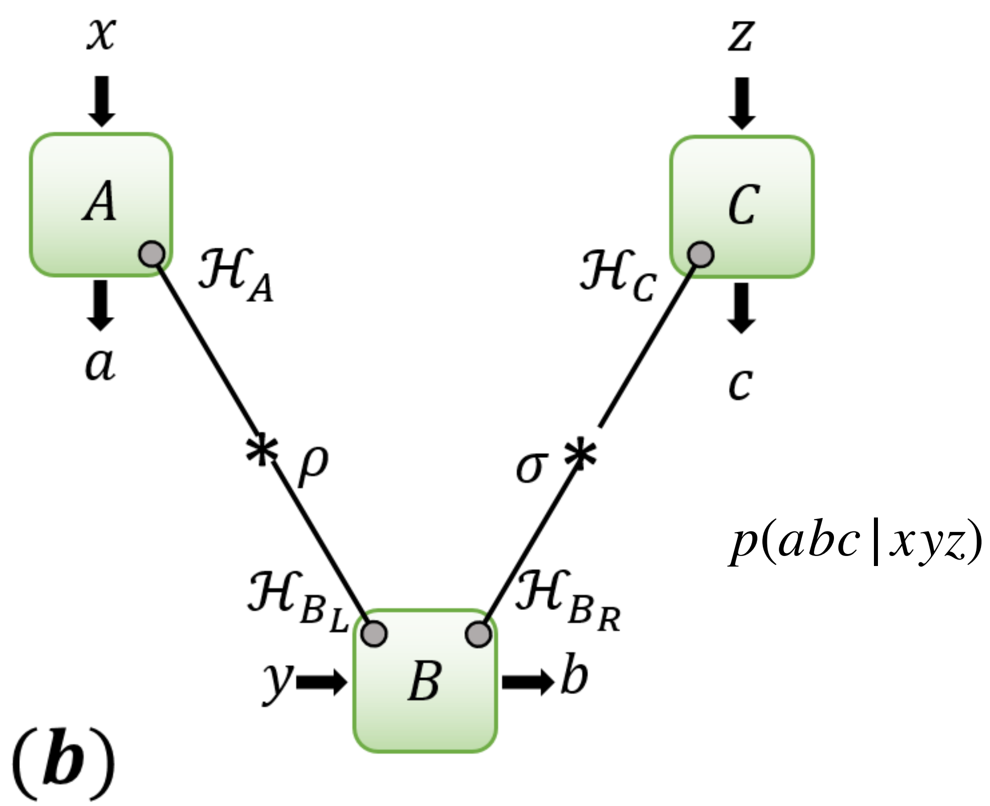
- Alice and Charlie are independent: $\mathrm{Tr}_\tau(A_{a|x}C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x})\mathrm{Tr}_\tau(C_{c|z})$, similarly for any products of $A_{a|x}, C_{c|z}$.

Bilocal scenario $Q_{bilocal}$ vs Bell scenario



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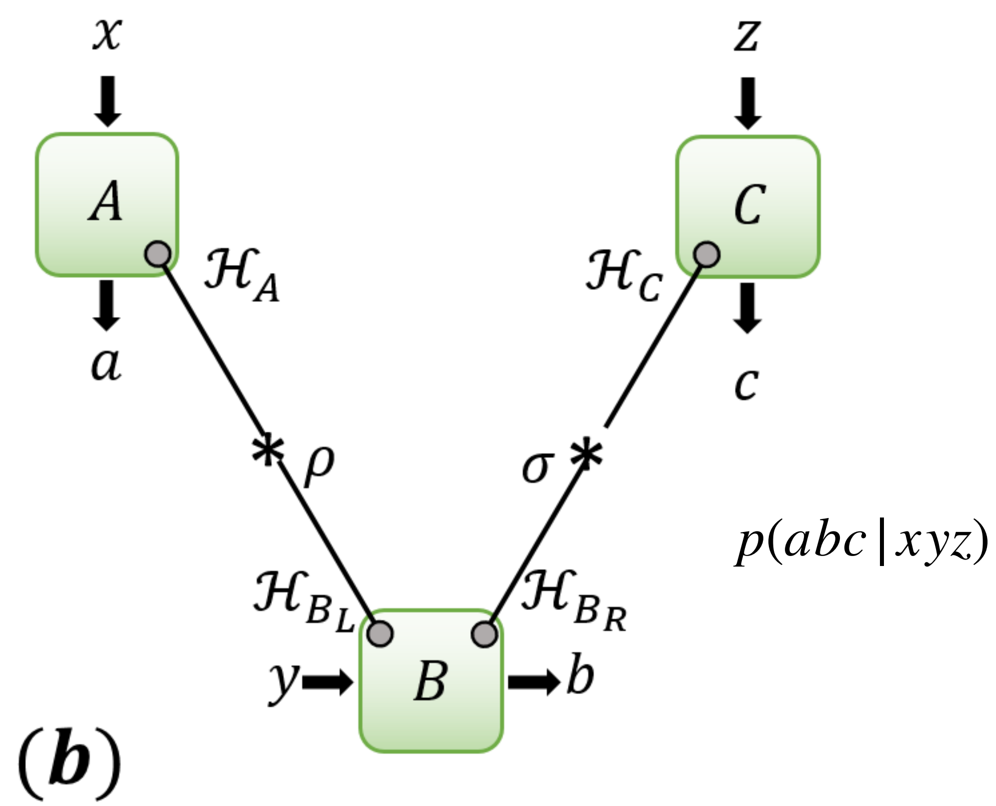

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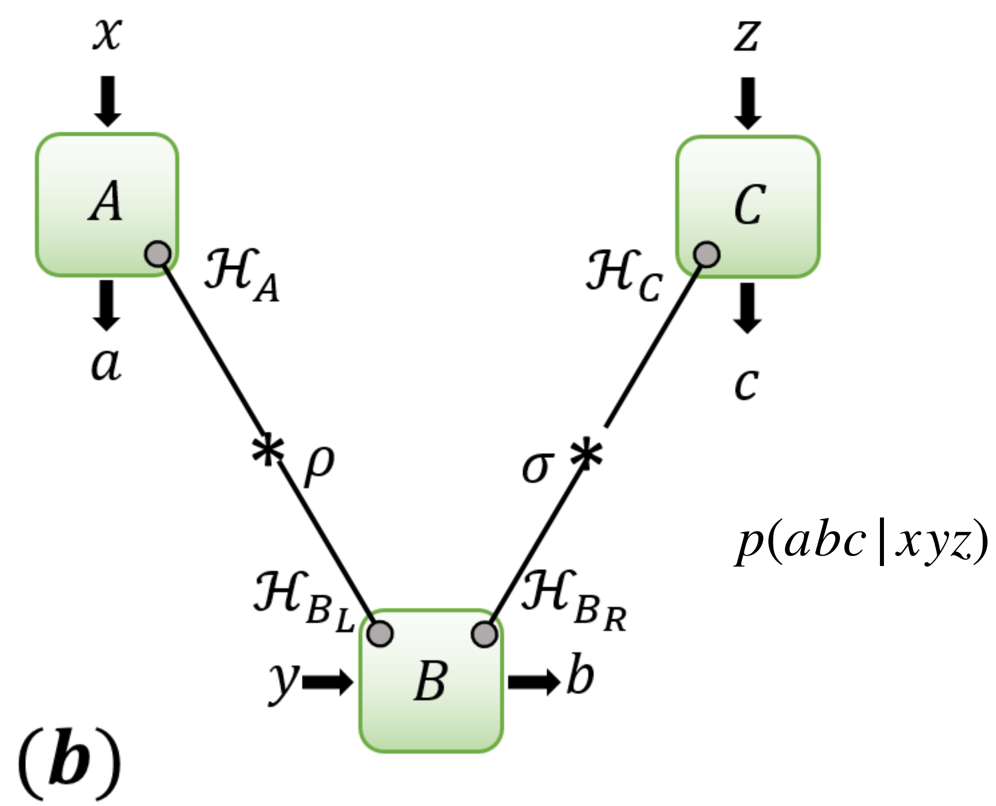

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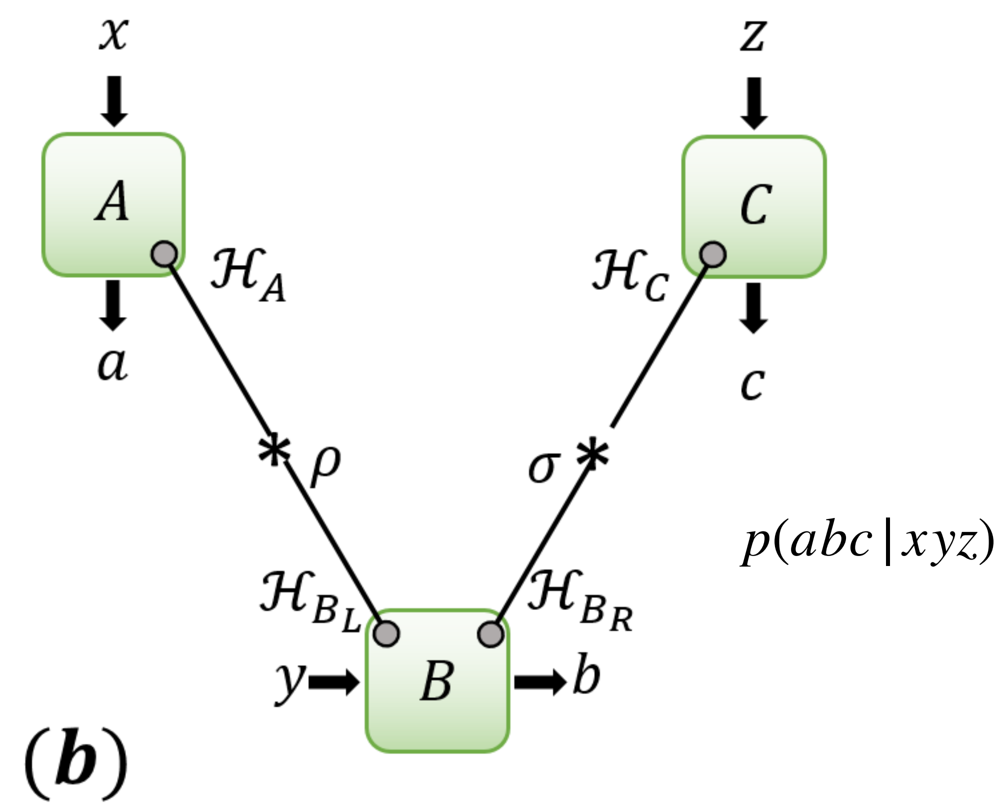
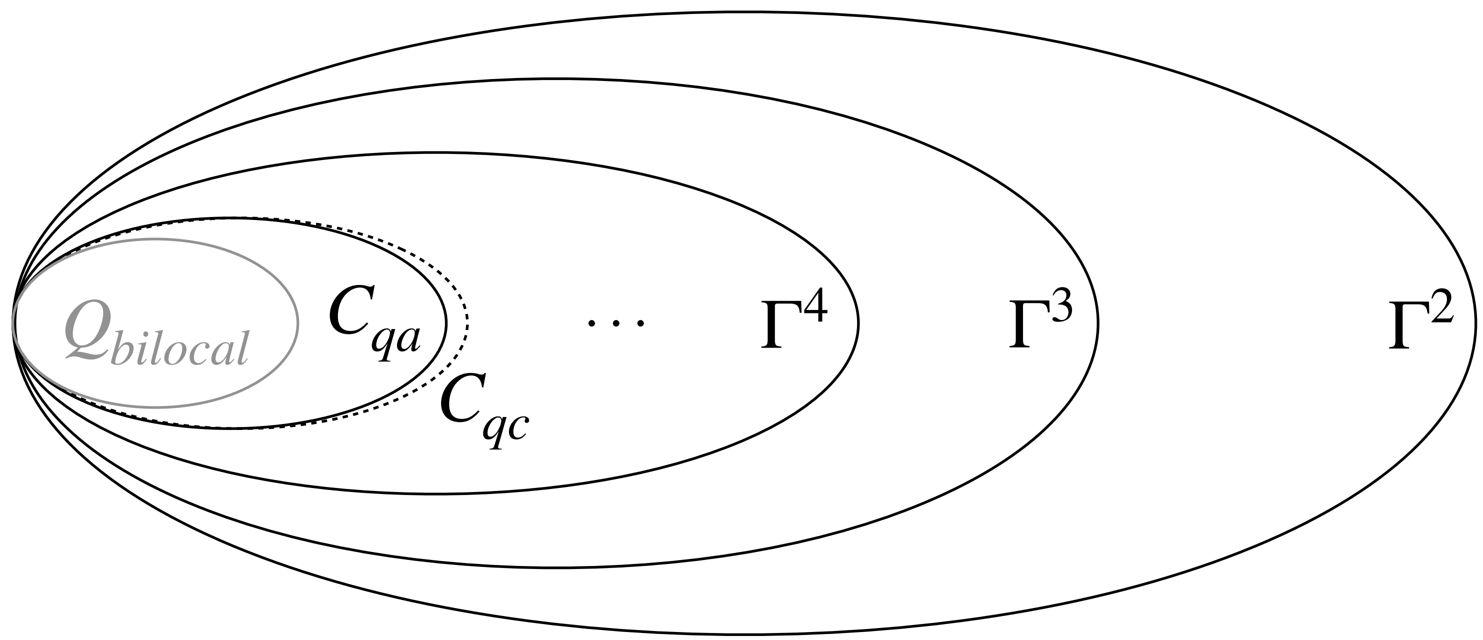


- We have $Q_{bilocal} \subsetneq C_{qa}$
- Bilocal scenario is always Bell (let $\tau = \rho \otimes \sigma$).
- Converse is not true, e.g. GHZ state cannot be separate. In fact, $Q_{bilocal}$ is not convex.

Bilocal scenario $Q_{bilocal}$ vs NPA hierarchy



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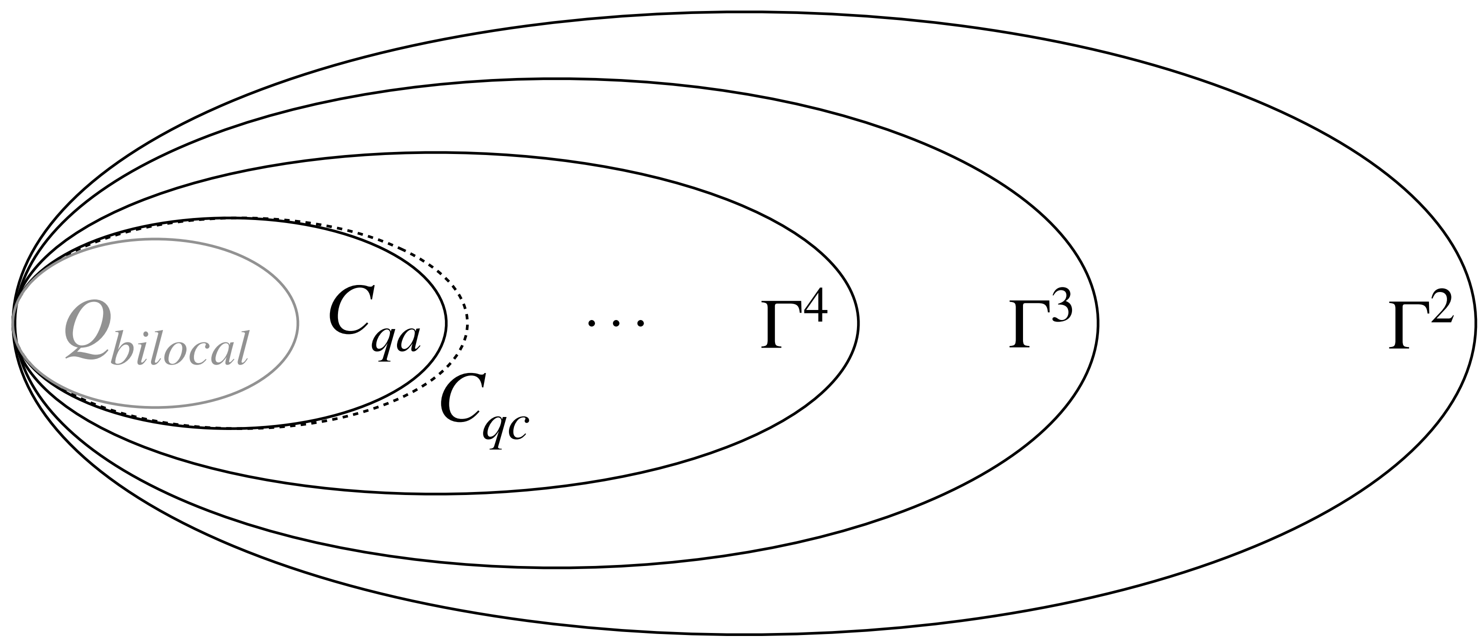

- Already an outer approximation to C_{qa}, standard NPA hierarchy is too unrestricted for $Q_{bilocal}$.

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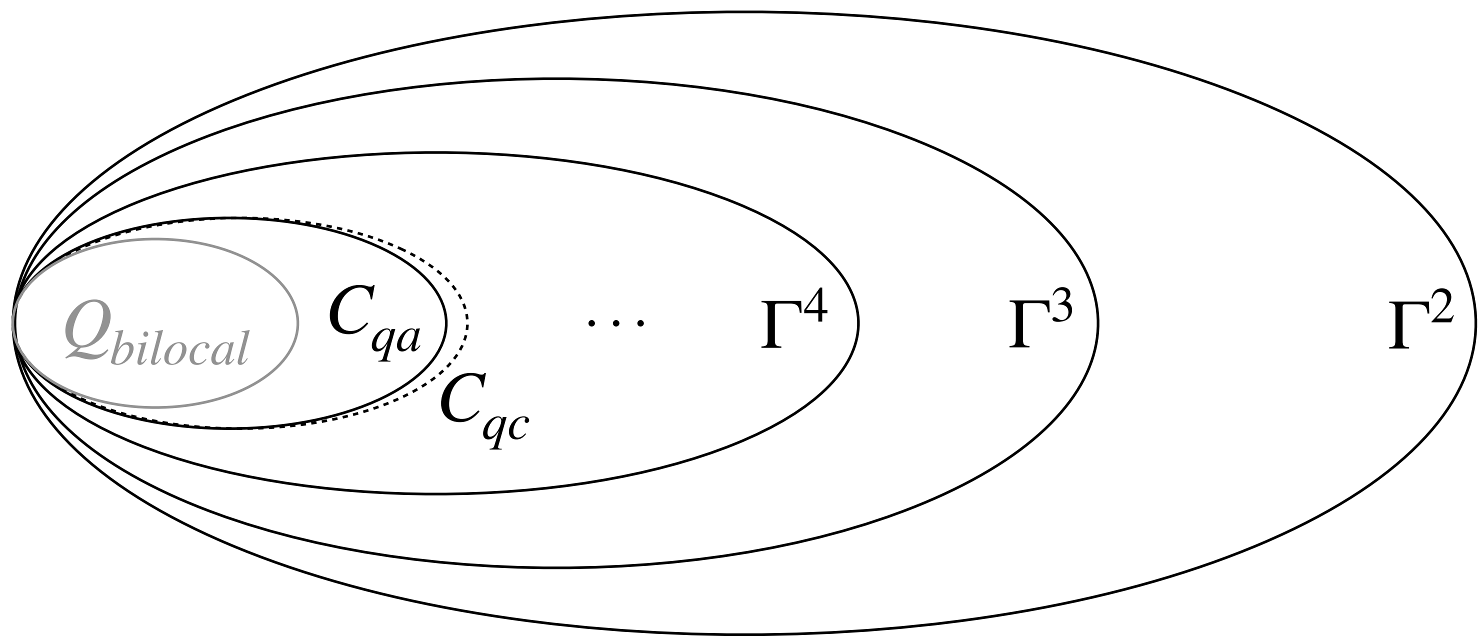

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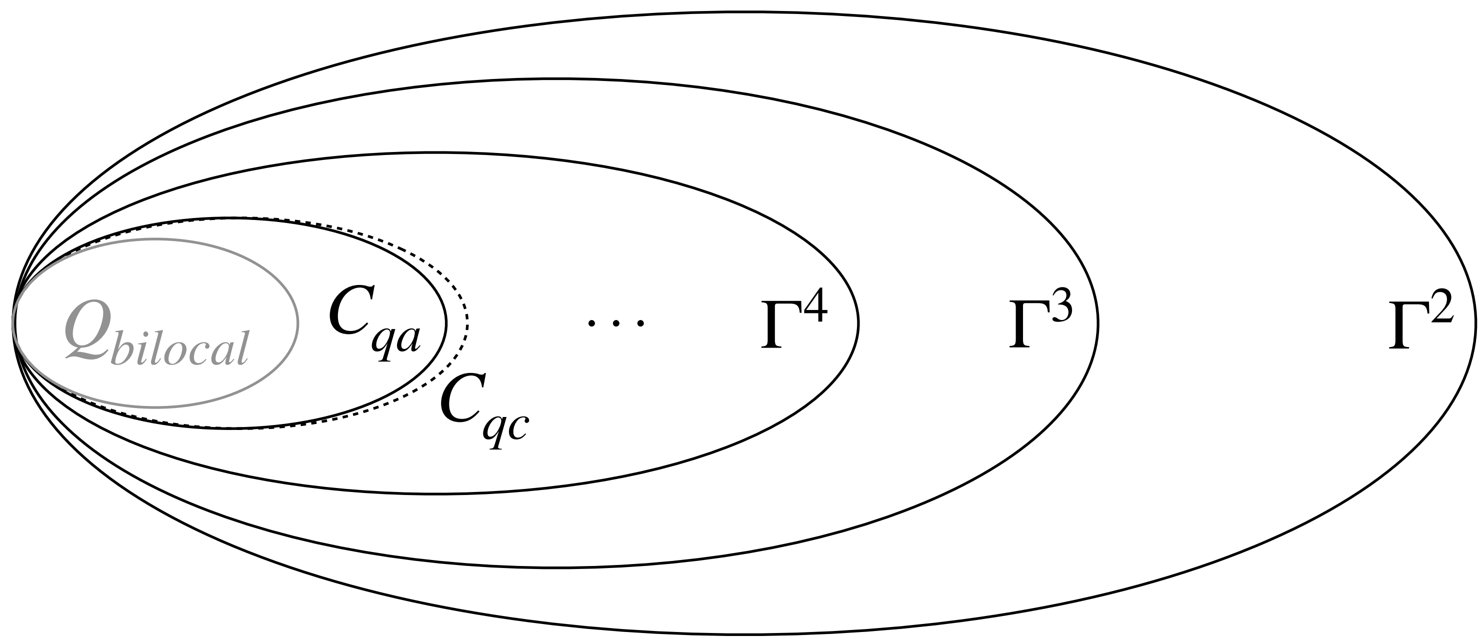

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- $\mathrm{Tr}_\tau(A_{a|x}C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x})\mathrm{Tr}_\tau(C_{c|z})$ and any product of A, C!

Factorisation bilocal hierarchy $\tilde{\Gamma}^n$

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\begin{array}{cc}
 & \begin{array}{ccccc} \mathbb{1} & A_{a|x} & B_{b|y} & C_{c|z} & \cdots \end{array} \\
\begin{array}{c} \mathbb{1} \\ (A_{a|x})^\dagger \\ (B_{b|y})^\dagger \\ (C_{c|z})^\dagger \\ (A_{a|x}A_{a'|x'})^\dagger \\ (A_{a|x}B_{b|y})^\dagger \\ \cdots \end{array} &
\begin{bmatrix}
1 & & & \mathrm{Tr}_\tau\left(C_{c|z}\right) & \\
\mathrm{Tr}_\tau\left(A_{a|x}\right) & & & & \\
 & & & & \\
 & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & & & \\
 & & & & \\
 & & & & \\
 & & & &
\end{bmatrix}
\end{array}
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- But for bilocal, also have factorisation constraints: e.g. $\tilde{\Gamma}^n_{A_{a|x},C_{c|z}} = \mathrm{Tr}_\tau(A_{a|x}C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x})\mathrm{Tr}_\tau(C_{c|z}) = \tilde{\Gamma}^n_{A_{a|x},1} \cdot \tilde{\Gamma}^n_{1,C_{c|z}}$ and arbitrary products.

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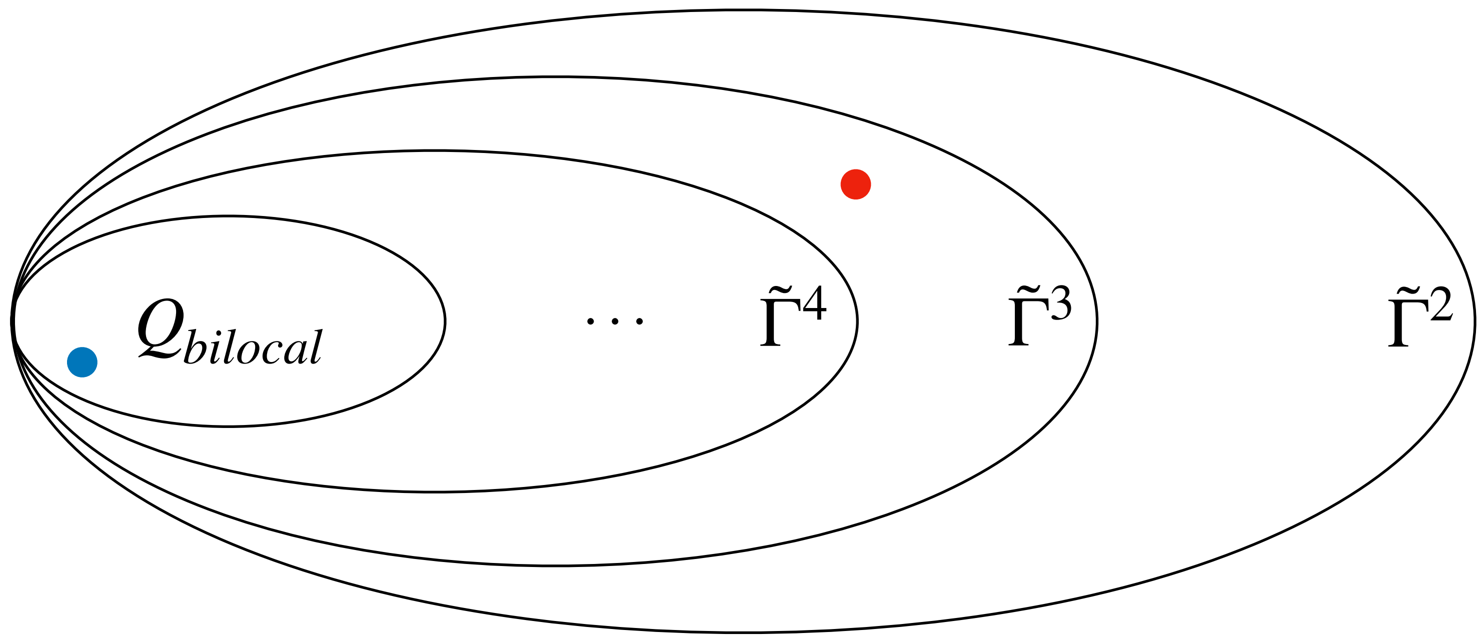
- Define $\tilde{\Gamma}^n = \Gamma^n +$ factorisation constraints

Factorisation hierarchy is necessary



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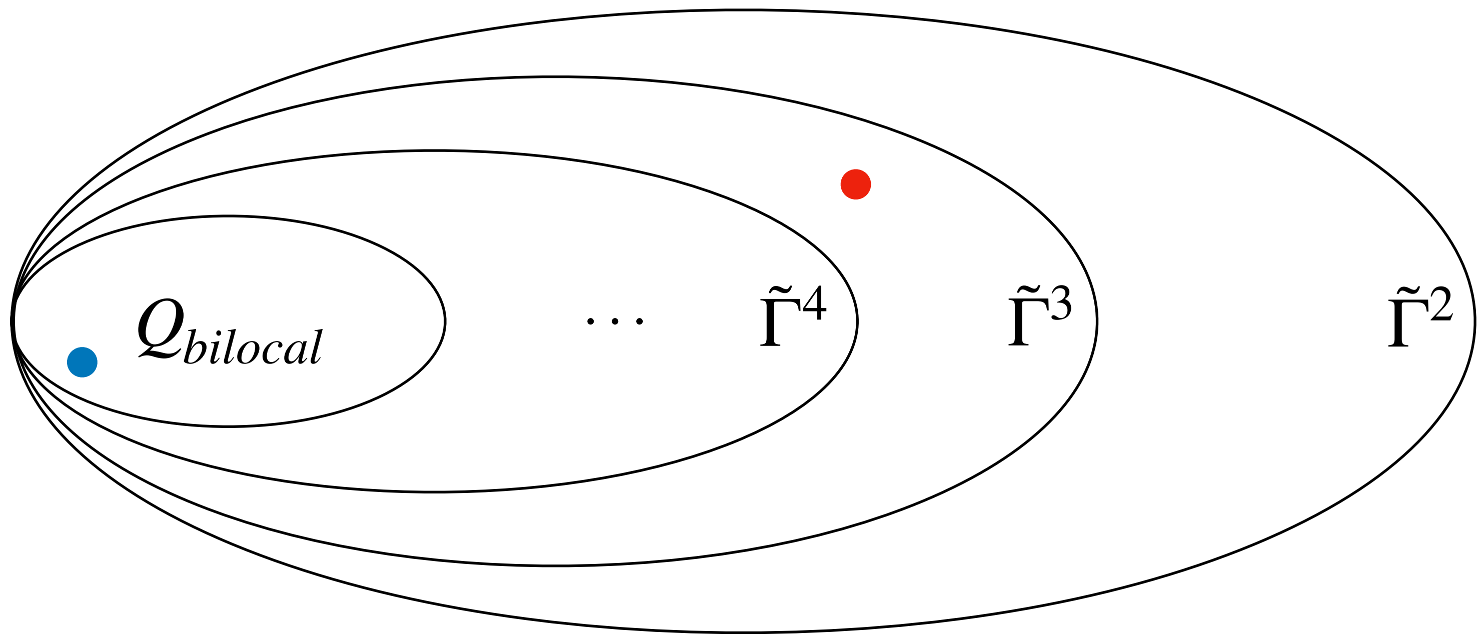

- New hierarchy $\tilde{\Gamma}^n = \Gamma^n$ + factorisation constraints.

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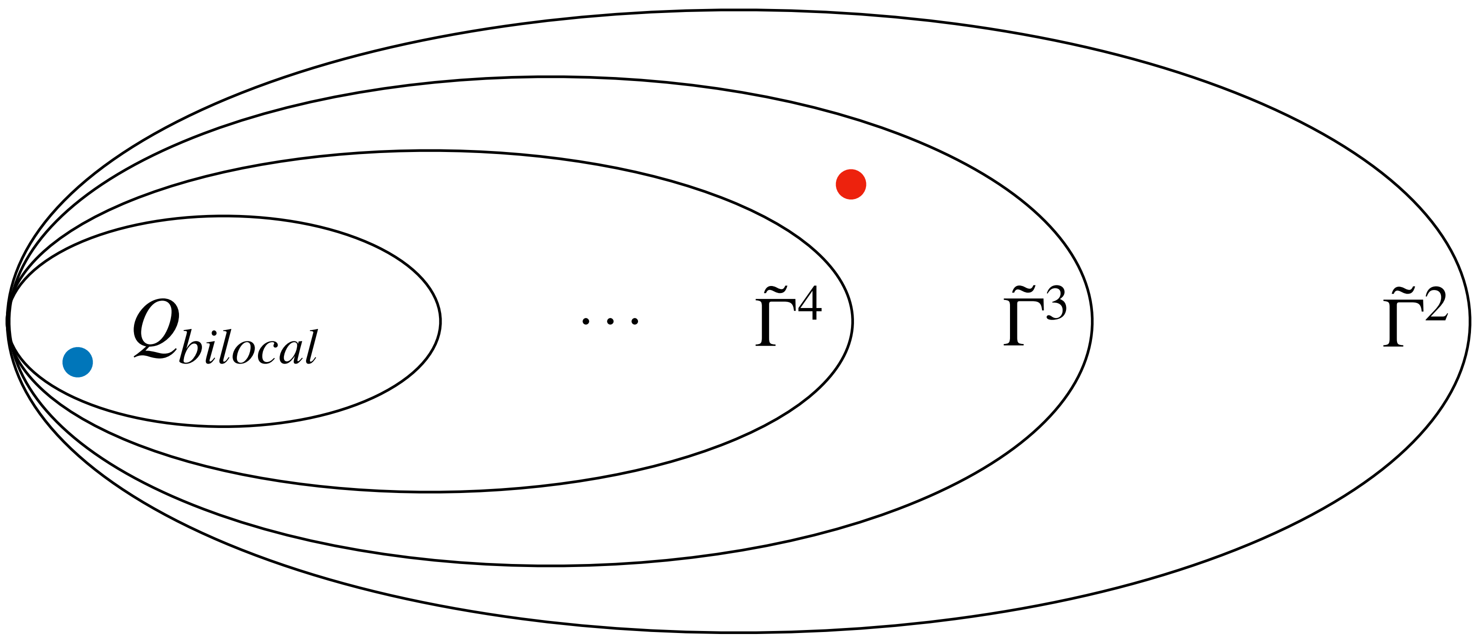

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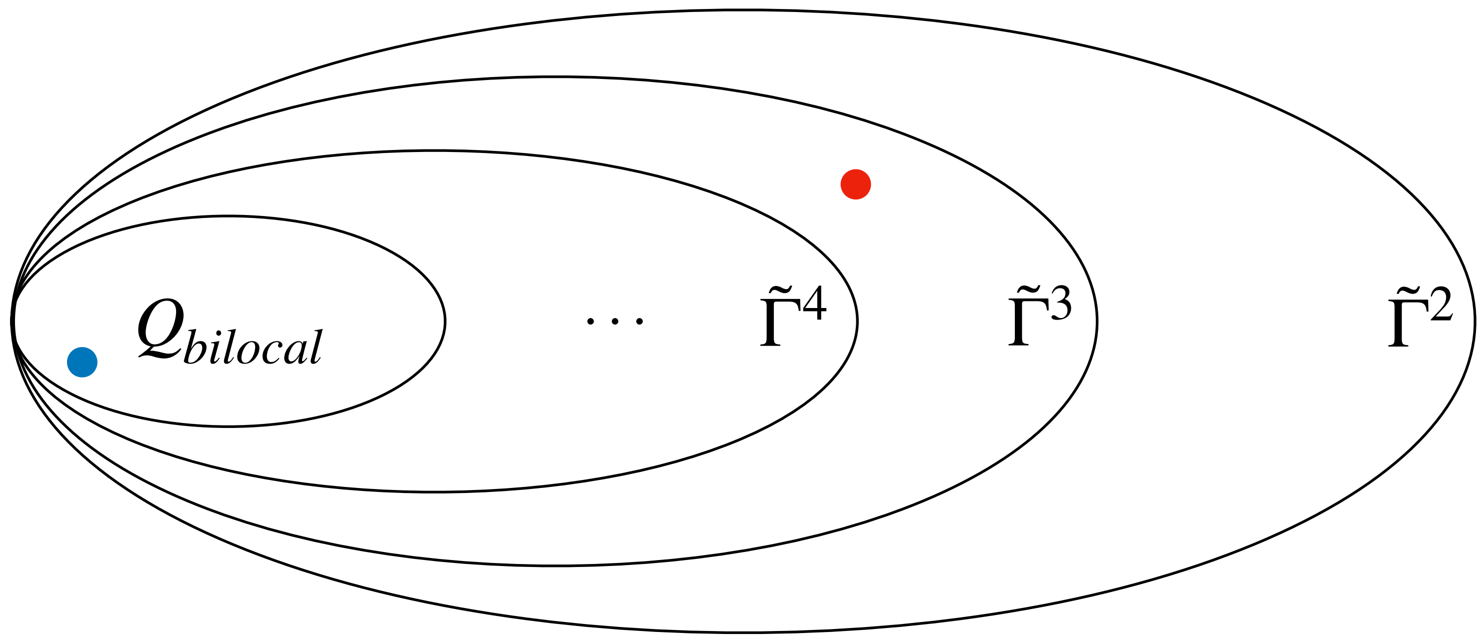

- New hierarchy $\tilde{\Gamma}^n$ = Γ^n + factorisation constraints.
- If $\overrightarrow{P} \in Q_{bilocal}$, then for all n there exists a compatible $\tilde{\Gamma}^n$.
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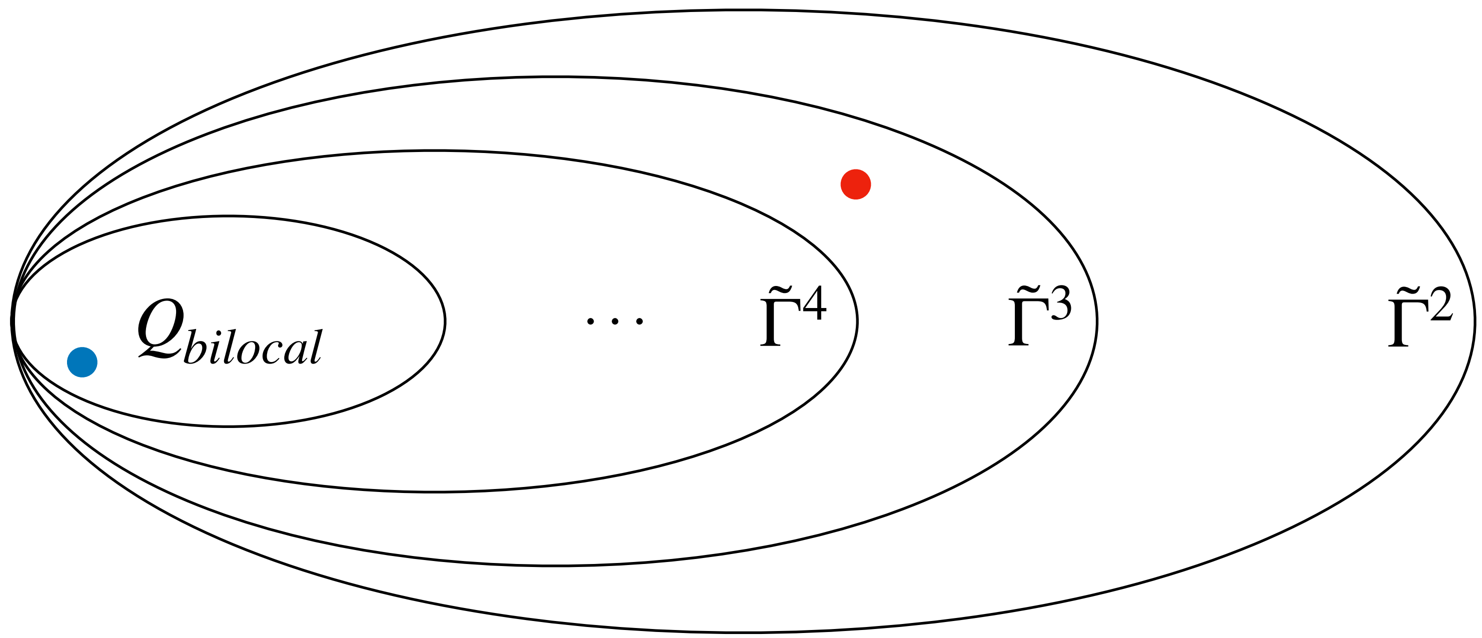

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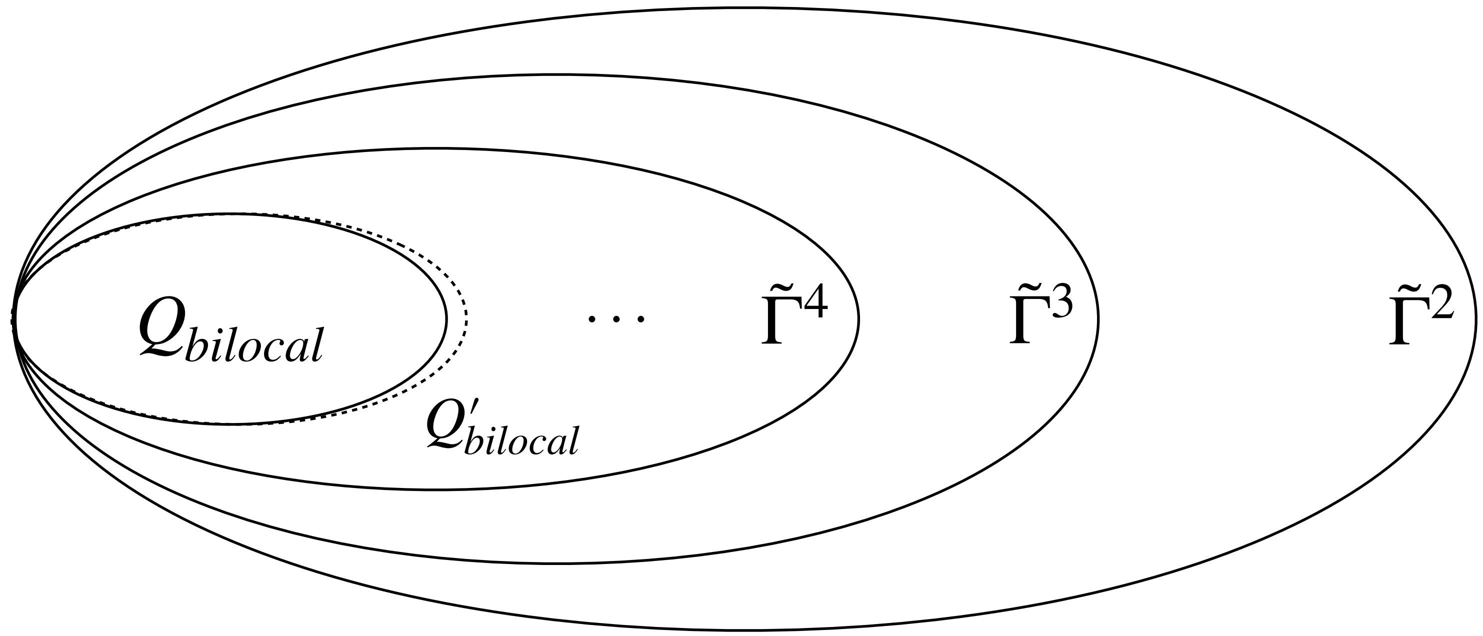
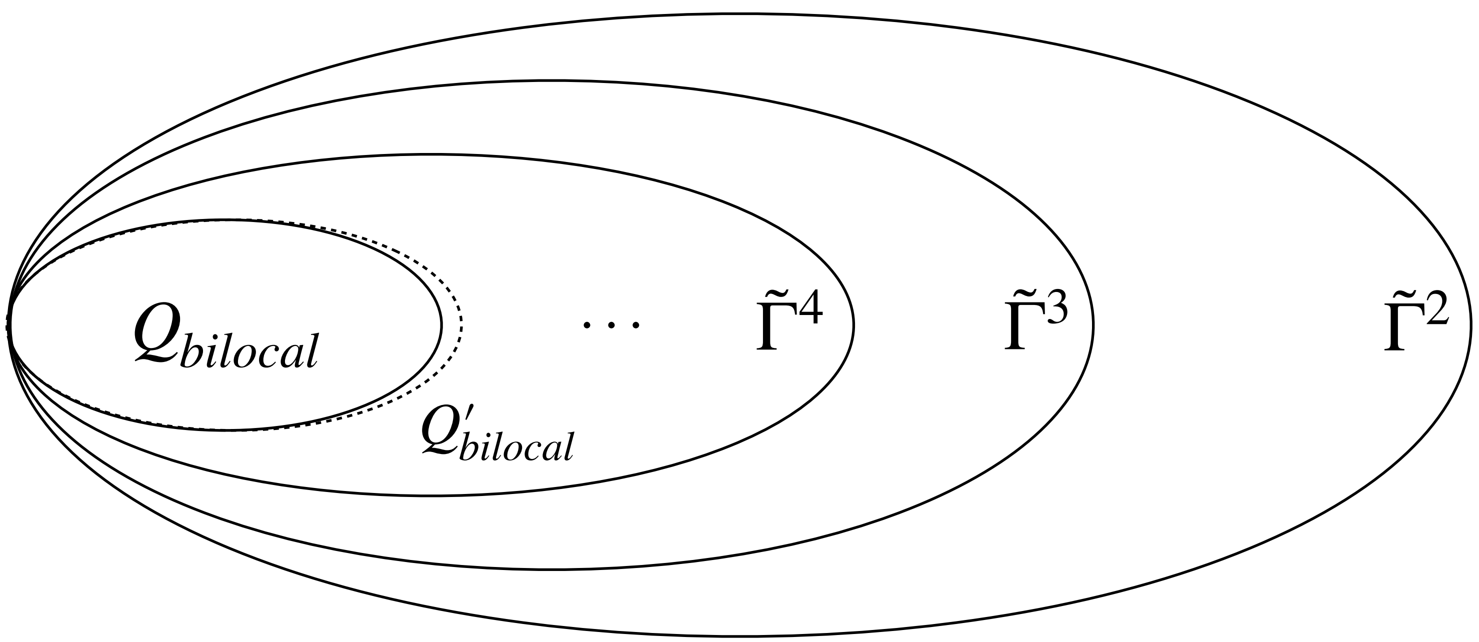
- But nonlinear, it is *not* SDP!

Is factorisation hierarchy sufficient?



Is factorisation hierarchy sufficient?

- What is $Q'_{bilocal} = lim_{n \to \infty} \tilde{\Gamma}^n$?



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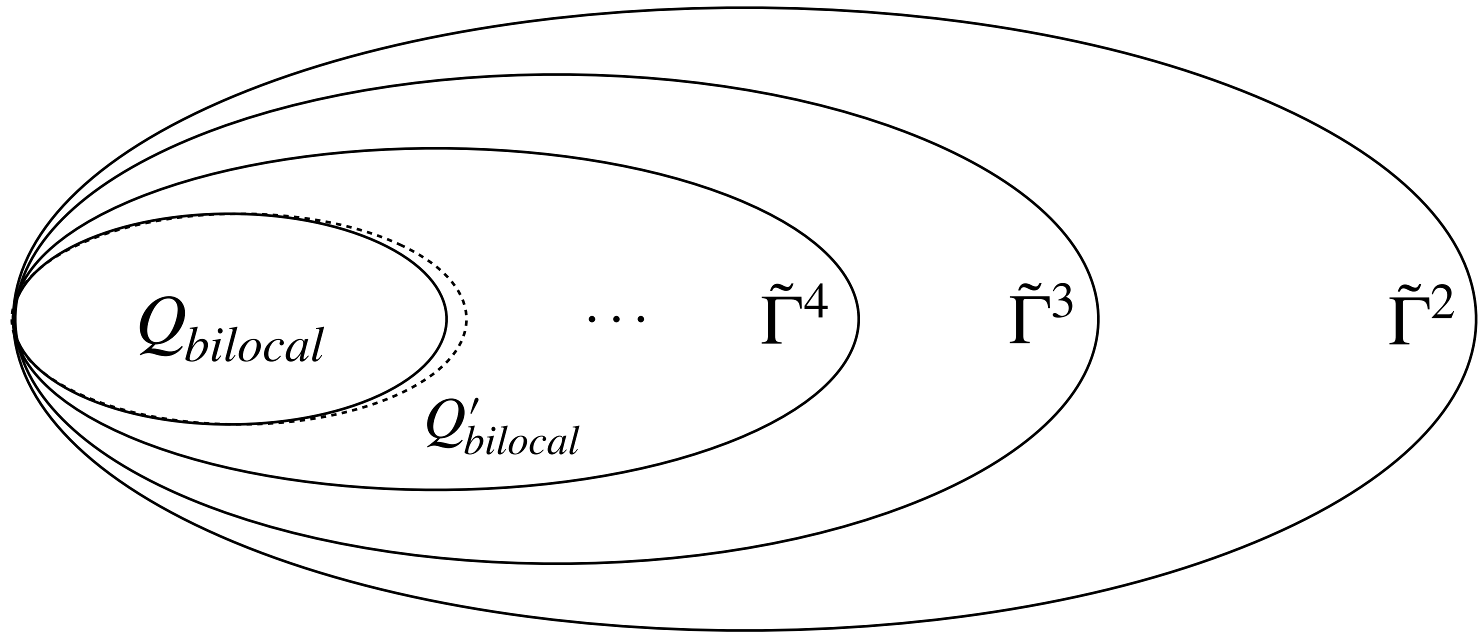
- What is $Q'_{bilocal} = lim_{n\to\infty}\tilde{\Gamma}^n$?
- Can we say $Q'_{bilocal} = Q_{bilocal}$? Analogous to C_{qa} vs C_{qc}?



Factorisation hierarchy: sufficiency

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    4. Projectors ρ, σ on H such that $\tau = \rho \cdot \sigma = \sigma \cdot \rho$;

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    4. Projectors ρ, σ on H such that $\tau = \rho \cdot \sigma = \sigma \cdot \rho$;
    5. $[A_{a|x}, \sigma] = [\rho, C_{c|z}] = 0$.

Factorisation hierarchy: sufficiency

- Main Theorem: If $\overrightarrow{P}$ admits $\tilde{\Gamma}^n$ for all $n \to \infty$, then $\overrightarrow{P} \in Q'_{bilocal}$ the *projector bilocal quantum distribution:*

    1. A global Hilbert space H with pure density operator τ;
    2. PVMs $\{A_{a|x}\}, \{B_{b|y}\}, \{C_{c|z}\}$ mutually commute;
    3. $p(abc\,|\,xyz) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z})$;
    4. Projectors ρ, σ on H such that $\tau = \rho \cdot \sigma = \sigma \cdot \rho$;
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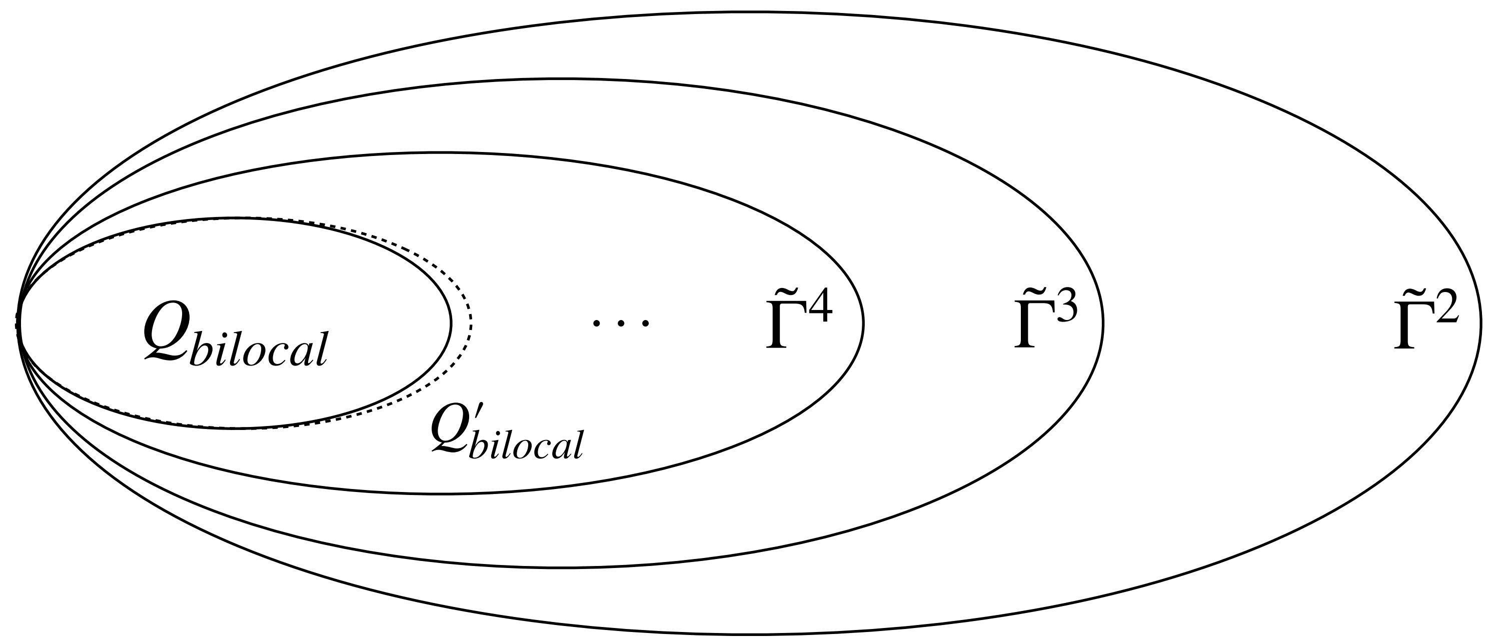
- Bilocal Tsirelson: [Ligthart and Gross, 2023], shows that $Q'_{bilocal}$ agrees $Q_{bilocal}$ in finite dimension

Factorisation bilocal hierarchy: summary



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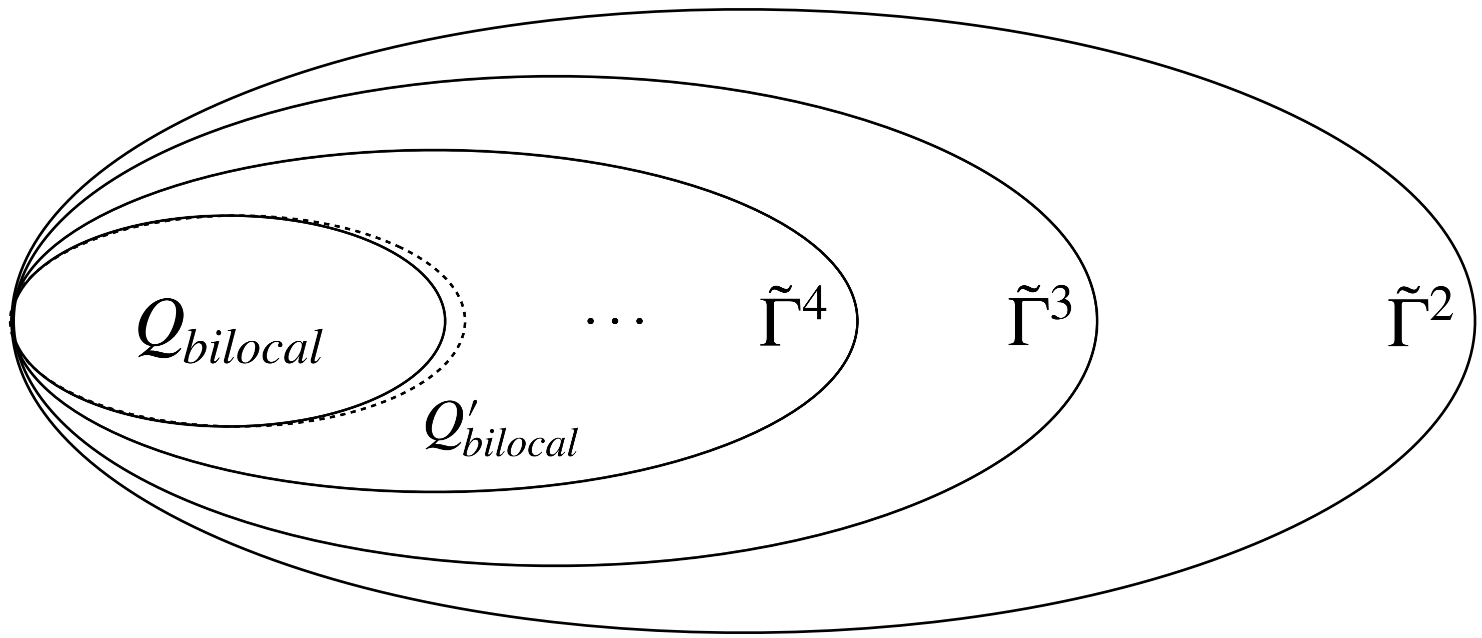
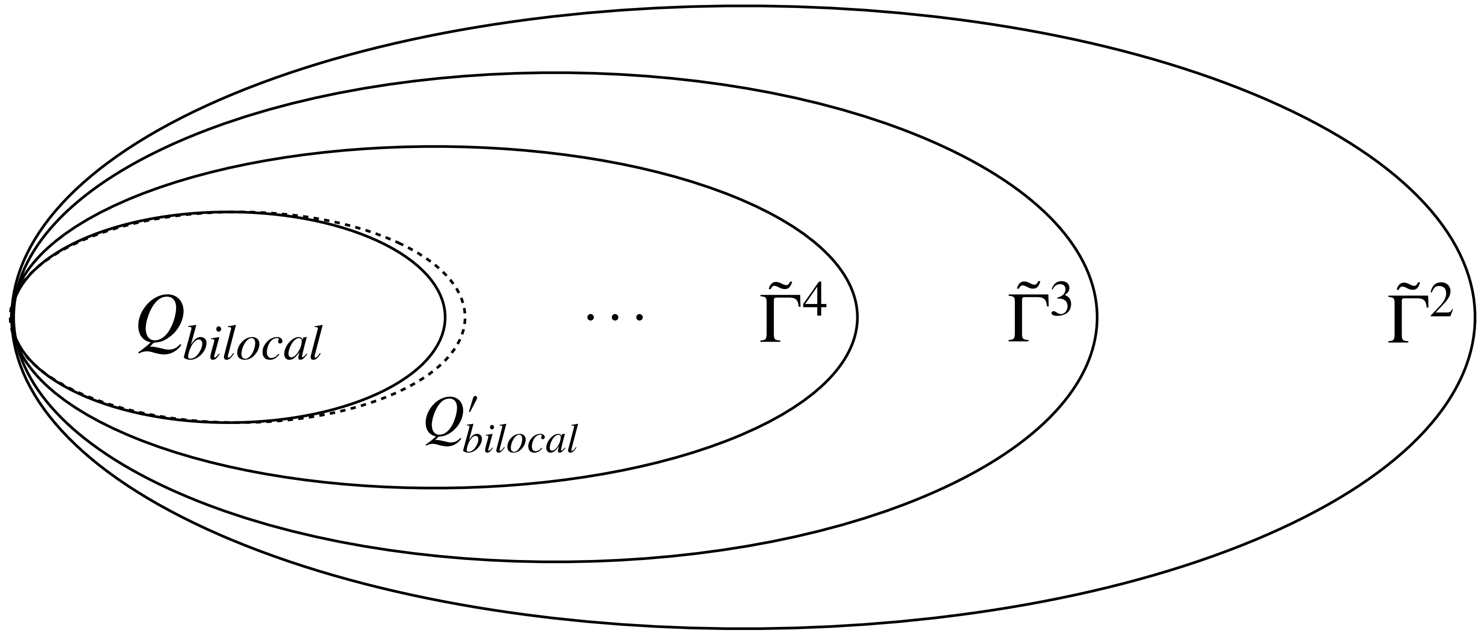


- A hierarchy $\tilde{\Gamma}^n$ converges to projector bilocal quantum $Q'_{bilocal}$ from the outside.

Factorisation bilocal hierarchy: summary



- A hierarchy $\tilde{\Gamma}^n$ converges to projector bilocal quantum $Q'_{bilocal}$ from the outside.

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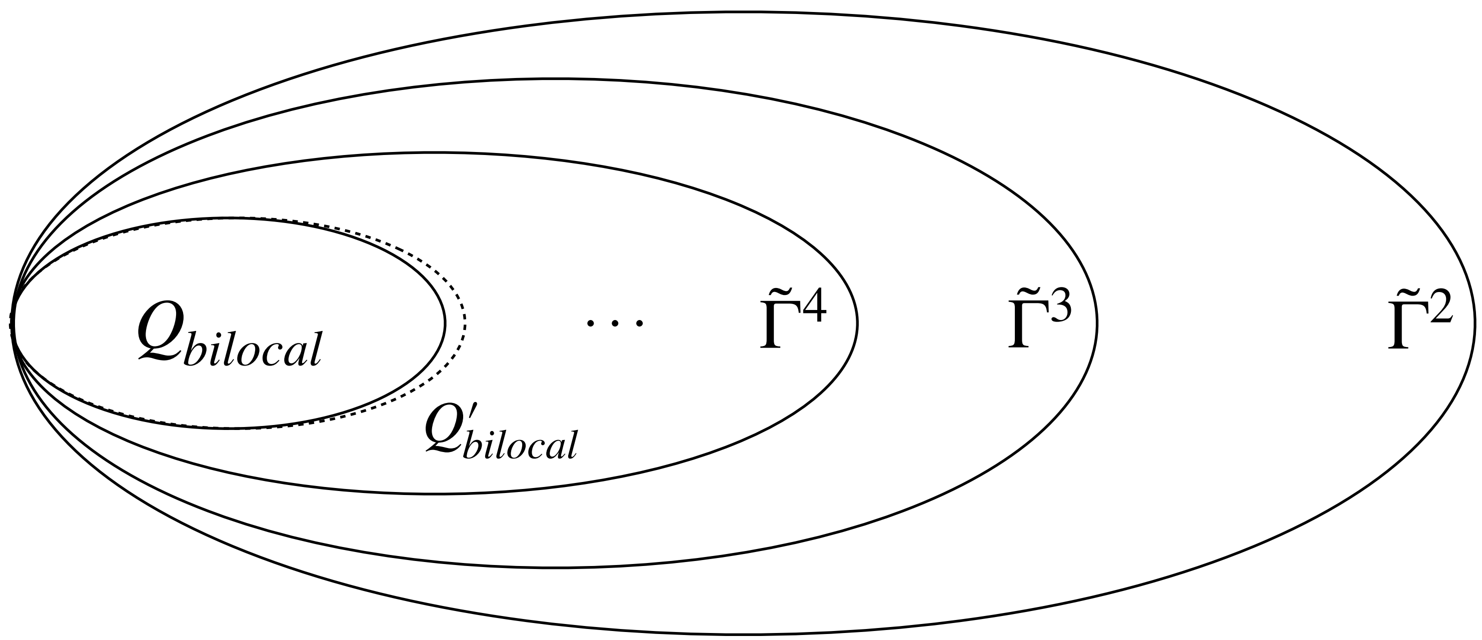

- A hierarchy $\tilde{\Gamma}^n$ converges to projector bilocal quantum $Q'_{bilocal}$ from the outside.

- $Q'_{bilocal}$ is equivalent to standard bilocal quantum model $Q_{bilocal}$ in finite dimension.

- Not SDP, cannot be solved by computers.

Scalar extension: linearise the hierarchy

Problem: factorisation constraints are not linear.

[Pozas-Kerstjens et al., 2019] introduces the original scalar extension:

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$$
\begin{array}{cc}
 & \begin{array}{cccccc} \mathbb{1} & A_{a|x} & \cdots & \kappa_{A_{a|x}} & \kappa_{A_{a|x}A_{a'|x'}} & \cdots \end{array} \\
\begin{array}{c} \mathbb{1} \\ (A_{a|x})^\dagger \\ (B_{b|y})^\dagger \\ (C_{c|z})^\dagger \\ (A_{a|x}A_{a'|x'})^\dagger \\ (A_{a|x}B_{b|y})^\dagger \\ \cdots \end{array} &
\begin{bmatrix} 1 & & & & & \\ & & & & & \\ & & & & & \\ & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & \cdots & \mathrm{Tr}_\tau\left(\kappa_{A_{a|x}}\hat{C}_{c|z}\right) & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}
\end{array}
$$

Scalar extension: first idea

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} \mathbb{1} & A_{a|x} & \cdots & \kappa_{A_{a|x}} & \kappa_{A_{a|x}A_{a'|x'}} & \cdots \end{array} \\
\begin{array}{c} \mathbb{1} \\ (A_{a|x})^\dagger \\ (B_{b|y})^\dagger \\ (C_{c|z})^\dagger \\ (A_{a|x}A_{a'|x'})^\dagger \\ (A_{a|x}B_{b|y})^\dagger \\ \cdots \end{array} & \begin{bmatrix} 1 & & & & \\ & & & & \\ & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & \cdots & \mathrm{Tr}_\tau\left(\kappa_{A_{a|x}}\hat{C}_{c|z}\right) & \\ & & & & \\ & & & & \end{bmatrix}
\end{array}
$$

Scalar extension: first idea

$$
\begin{array}{c|c}
 & \begin{matrix} \mathbb{1} & A_{a|x} & \cdots & \kappa_{A_{a|x}} & \kappa_{A_{a|x}A_{a'|x'}} & \cdots \end{matrix} \\
\begin{matrix} \mathbb{1} \\ (A_{a|x})^\dagger \\ (B_{b|y})^\dagger \\ (C_{c|z})^\dagger \\ (A_{a|x}A_{a'|x'})^\dagger \\ (A_{a|x}B_{b|y})^\dagger \\ \cdots \end{matrix} &
\begin{bmatrix} 1 & & & & & \\ & & & & & \\ & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & \cdots & \mathrm{Tr}_\tau\left(\kappa_{A_{a|x}}\hat{C}_{c|z}\right) & & \\ & & & & & \\ & & & & & \end{bmatrix}
\end{array}
$$

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1 & & & & & \\
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 & & & & & \\
 & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & \cdots & \mathrm{Tr}_\tau\left(\kappa_{A_{a|x}}\hat{C}_{c|z}\right) & & \\
 & & & & & \\
 & & & & & \\
 & & & & &
\end{bmatrix}
\end{array}$$

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 & & & & & \\
 & & & & & \\
 & & & & &
\end{bmatrix}
\end{array}
$$

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 & & & & & \\
 & & & & & \\
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\end{array}$$

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$$
\begin{array}{c}
 \\
\mathbb{1} \\
(A_{a|x})^\dagger \\
(B_{b|y})^\dagger \\
(C_{c|z})^\dagger \\
(A_{a|x}A_{a'|x'})^\dagger \\
(A_{a|x}B_{b|y})^\dagger \\
\cdots
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
\mathbb{1} & A_{a|x} & \cdots & \kappa_{A_{a|x}} & \kappa_{A_{a|x}A_{a'|x'}} & \cdots
\end{array} \\
\begin{bmatrix}
1 & & & & & \\
 & & & & & \\
 & & & & & \\
 & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & \cdots & \mathrm{Tr}_\tau\left(\kappa_{A_{a|x}}\hat{C}_{c|z}\right) & & \\
 & & & & & \\
 & & & & & \\
 & & & & &
\end{bmatrix}
\end{array}
$$

- New commutative variables to Alice: $\kappa_{A_{a|x}} \cancel{=} \mathrm{Tr}_\tau(A_{a|x})\mathrm{Id}$

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 \\
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\cdots
\end{array}
\begin{array}{c}
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1 & & & & & \\
 & & & & & \\
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 & & & & & \\
 & & & & & \\
 & & & & &
\end{bmatrix}
\end{array}
$$

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But we have a fix!

- Then
$\mathrm{Tr}_\tau(\kappa_{A_{a|x}} C_{c|z}) = \mathrm{Tr}_\tau(\mathrm{Tr}_\tau(A_{a|x})\mathrm{Id} \cdot C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x}) \cdot \mathrm{Tr}_\tau(C_{c|z})$

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Scalar extension: on the carpet

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \mathbb{1} & A_{a|x} & \cdots & \kappa_{A_{a|x}} & \kappa_{A_{a|x}A_{a'|x'}} & \cdots \end{array} \\
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\begin{bmatrix} 1 & & & & & \\ & & & & & \\ & & & & & \\ & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & \cdots & \mathrm{Tr}_\tau\left(\kappa_{A_{a|x}}\hat{C}_{c|z}\right) & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}
\end{array}
$$

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\end{array}
$$

- Need new commutative variables: $\kappa_{A_{a|x}}$, $\kappa_{A_{a|x}B_{b|y}}$, $\kappa_{A_{a|x}A_{a'|x'}C_{c|z}} \cdots$

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 & & & & & \\
 & & & & & \\
 & & & & &
\end{bmatrix}
\end{array}$$

- Need new commutative variables: $\kappa_{A_{a|x}}$, $\kappa_{A_{a|x}B_{b|y}}$, $\kappa_{A_{a|x}A_{a'|x'}C_{c|z}}\cdots$
- Need more complicated constraints

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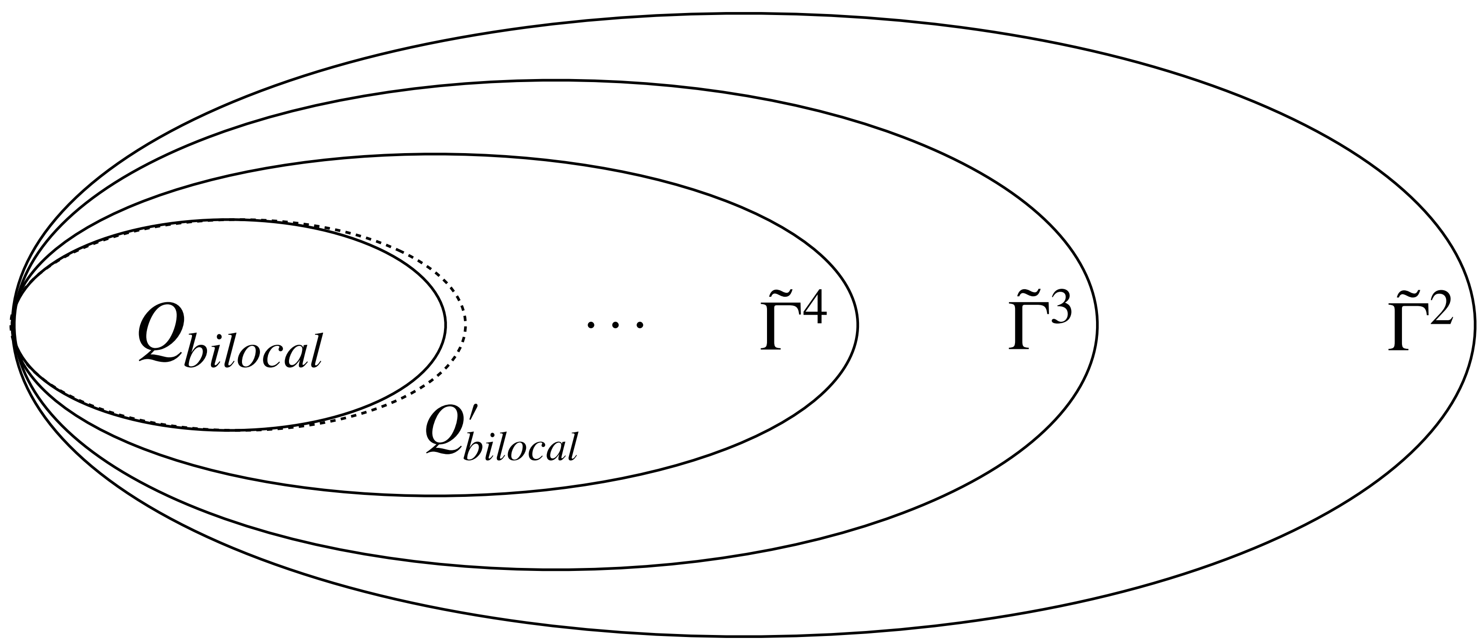
\end{bmatrix}
\end{array}
$$

- Need new commutative variables: $\kappa_{A_{a|x}}, \kappa_{A_{a|x}B_{b|y}}, \kappa_{A_{a|x}A_{a'|x'}C_{c|z}} \cdots$
- Need more complicated constraints
- A convergent scalar extension hierarchy Ω^n to $Q'_{bilocal}$

Bilocal network: summary on two hierarchies



- Two hierarchies, $\tilde{\Gamma}^n$ and Ω^n, both converge to projector commutator model $Q'_{bilocal}$ from the outside.

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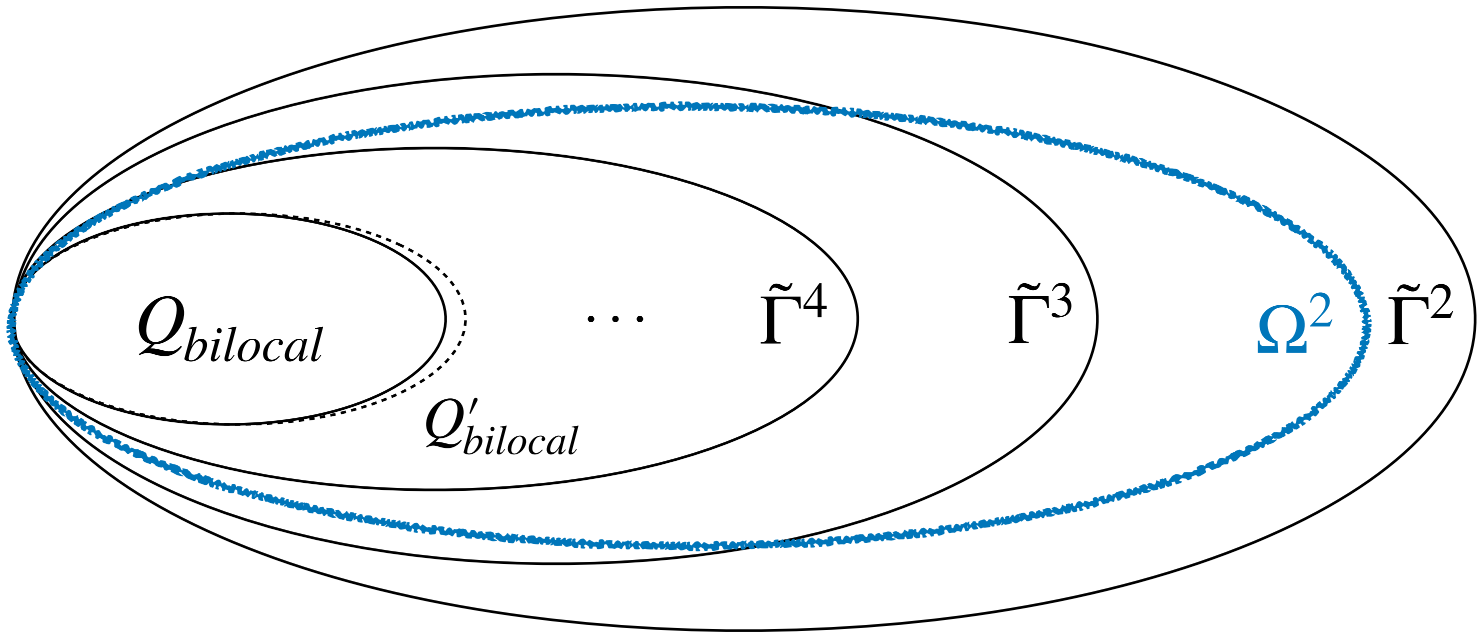

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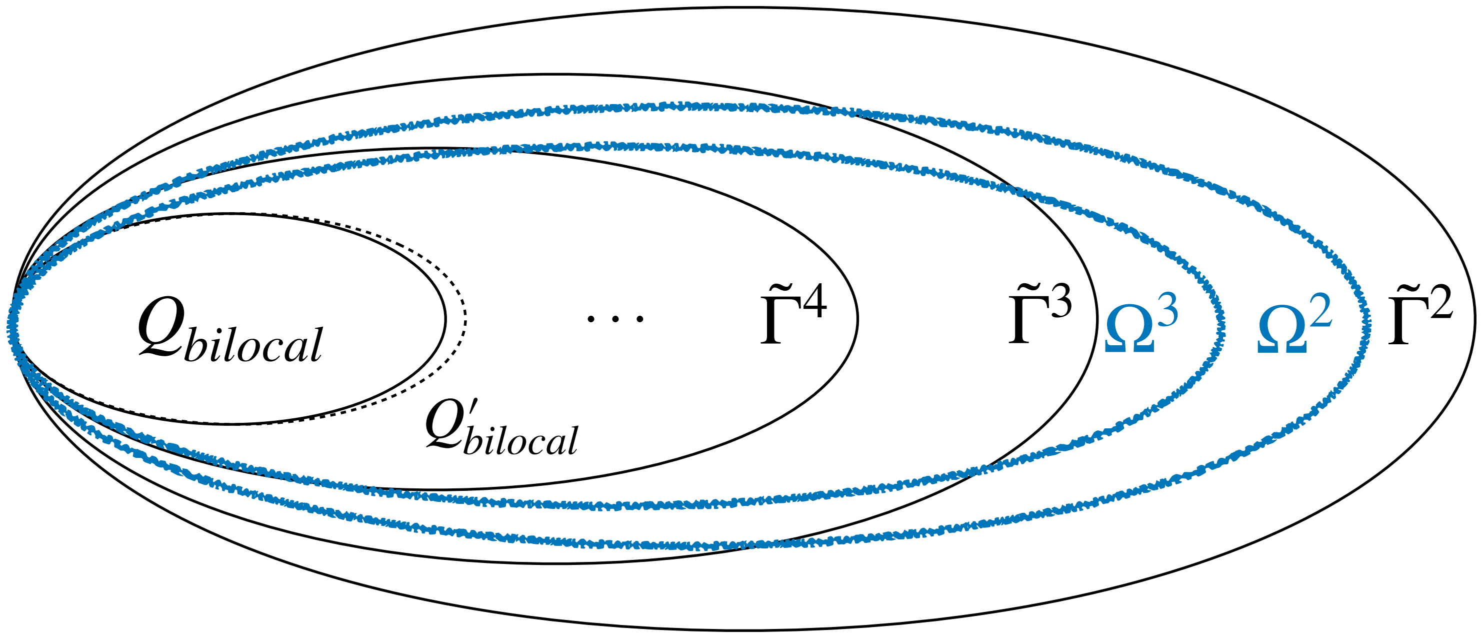

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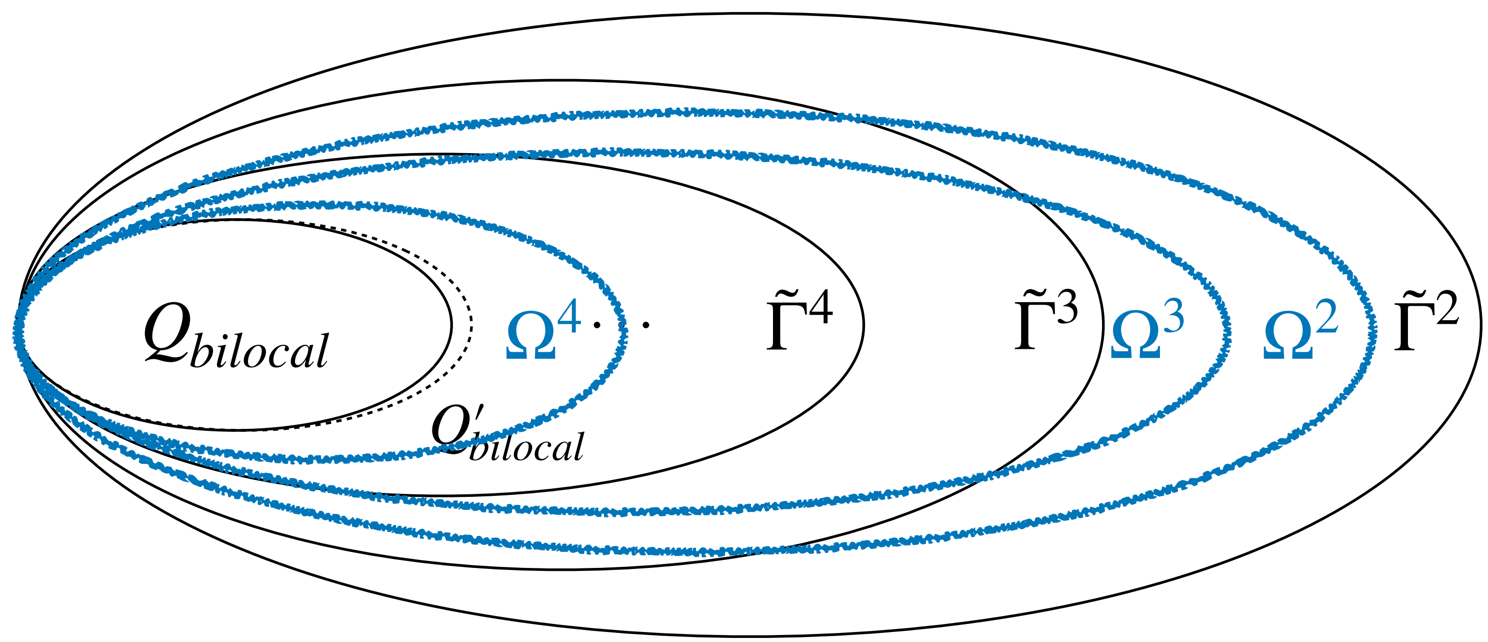

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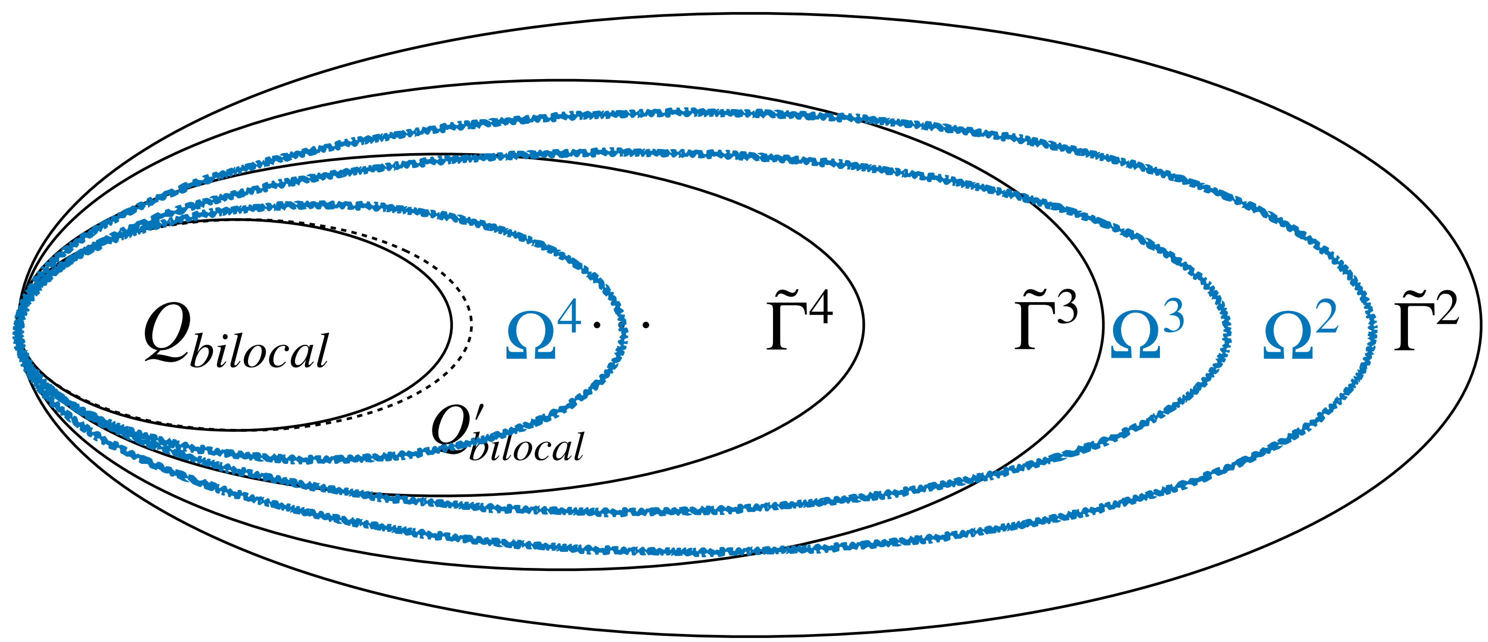

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- In finite dimension, they both characterise the standard QM bilocal network correlations.

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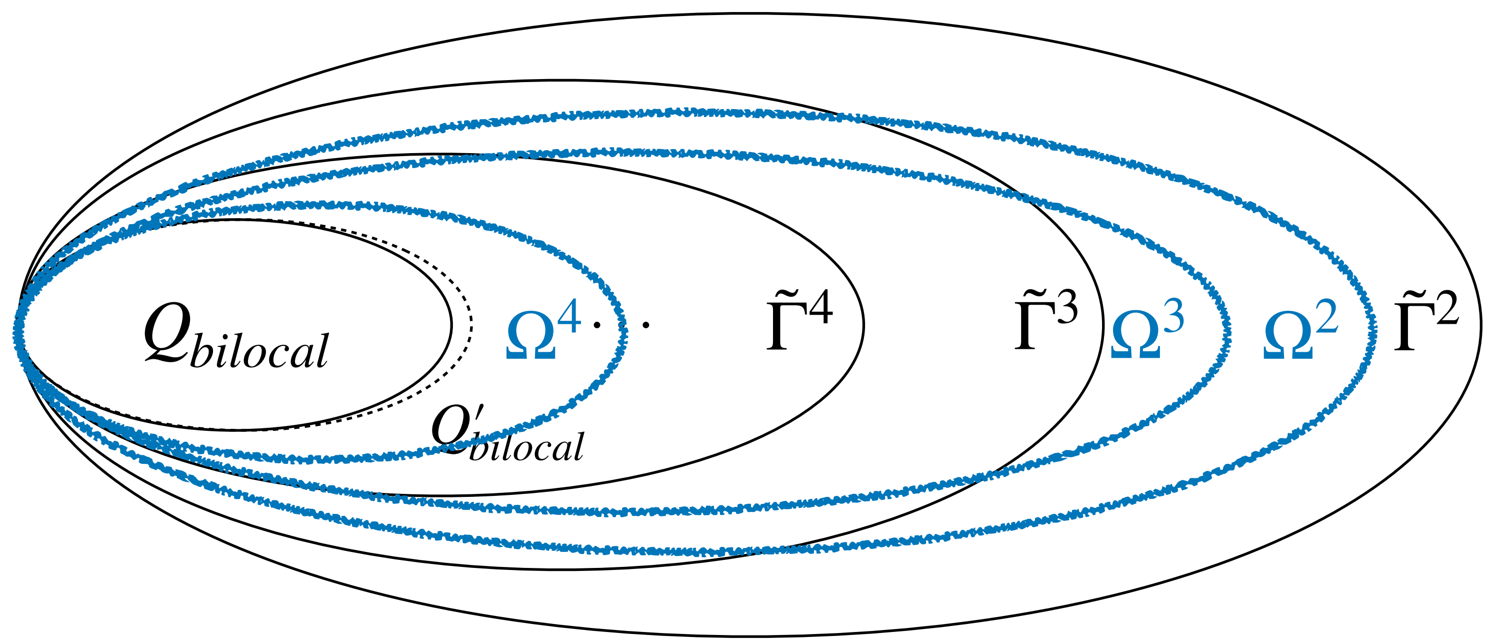

- Two hierarchies, $\tilde{\Gamma}^n$ and Ω^n, both converge to projector commutator model $Q'_{bilocal}$ from the outside.
- In finite dimension, they both characterise the standard QM bilocal network correlations.
- Scalar extension hierarchy Ω^n is SDP.

Bilocal network: the wider context

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- Together, bilocal network scenario (actually, stars) can be completely characterized in the C^*-algebraic/Heisenberg picture.
- But, more general networks remain open. Quantum inflation [Wolfe et al., 2021]?