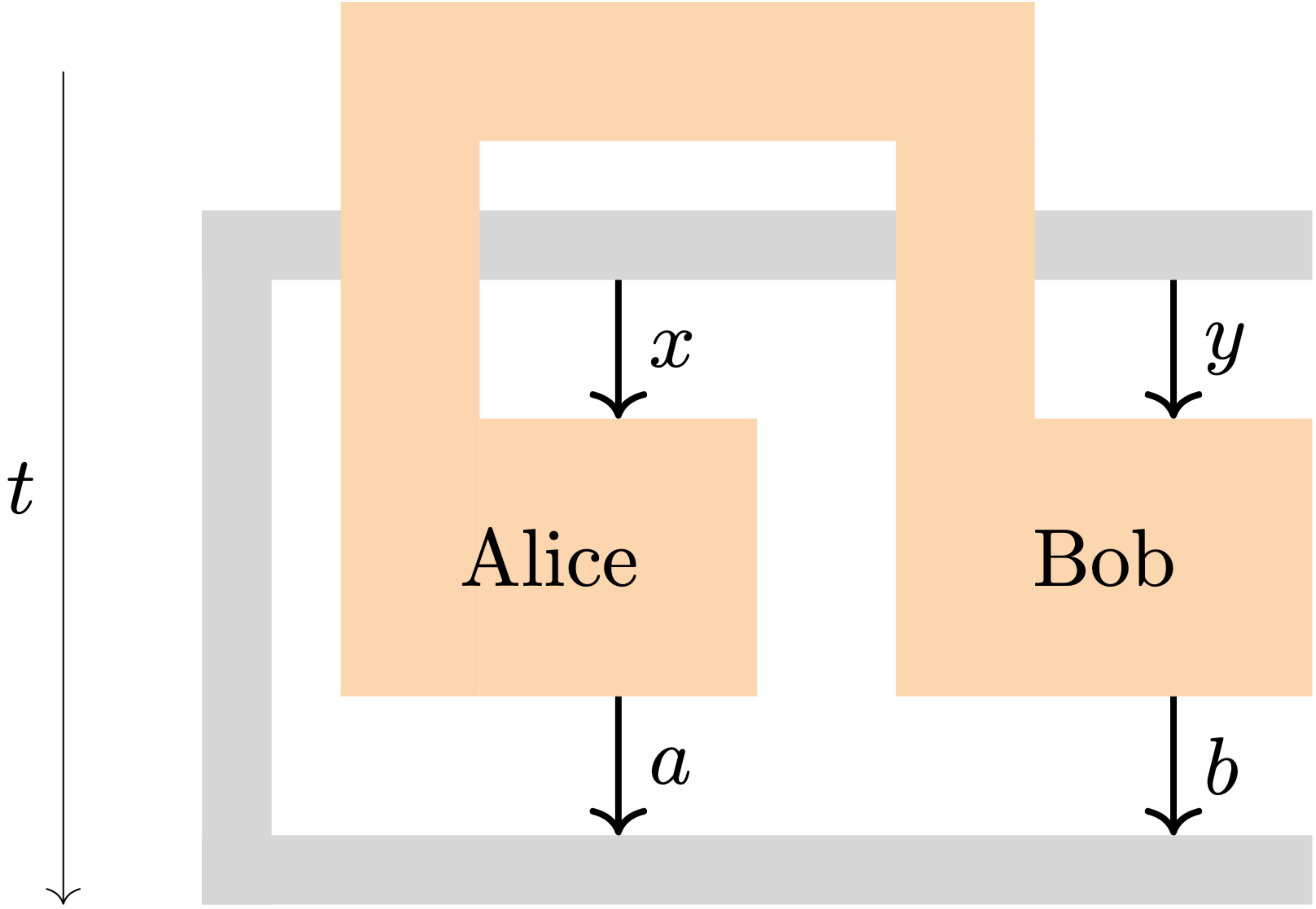
Quantum soundness for compiled Bell games

(1) Quantitatively for all bipartite games
(2) Asymptotically for all multipartite games

Joint talk by Matilde Baroni and Xiangling Xu

17/07/2025, IWOTA25 Twente

Non-locality 101



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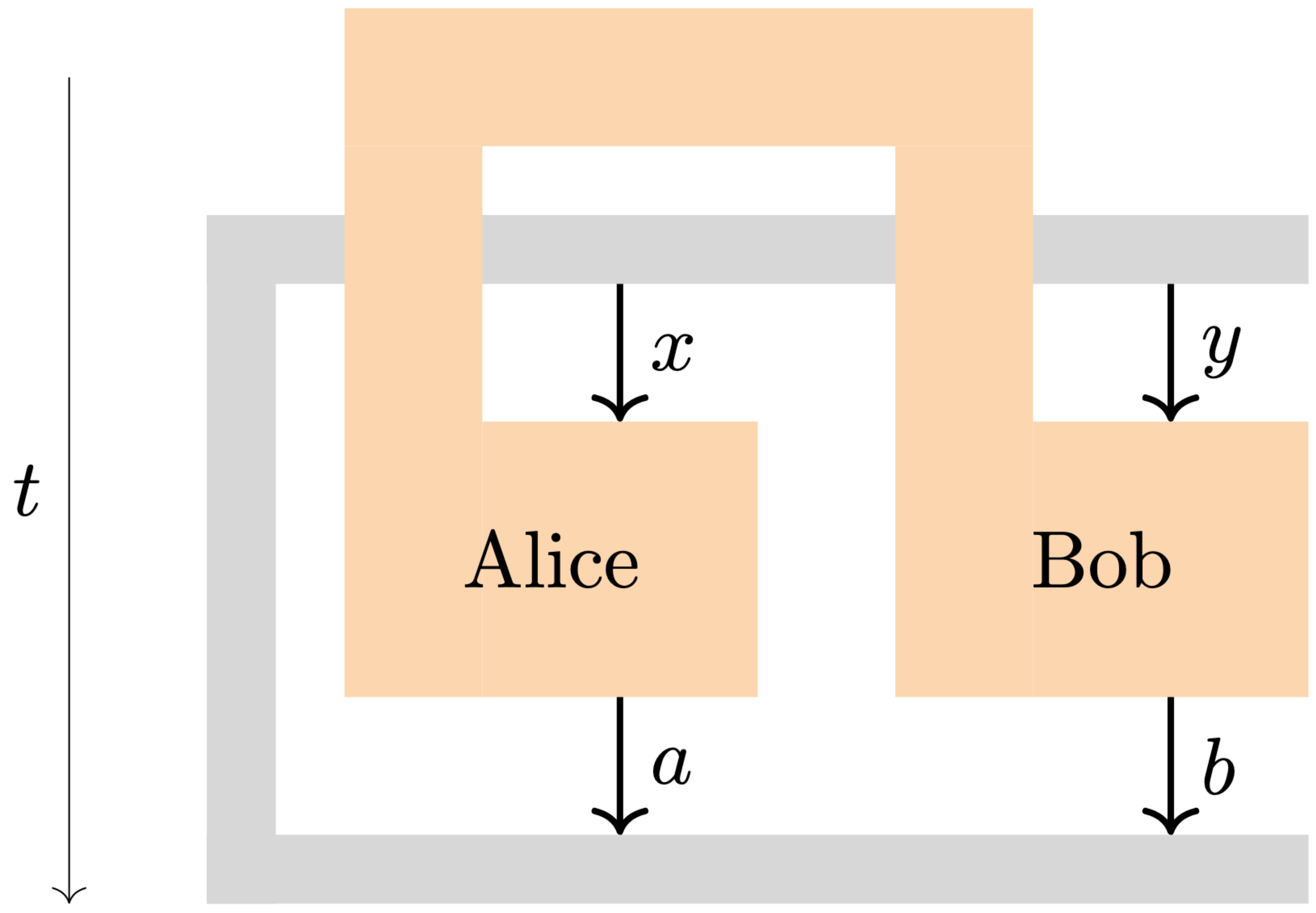

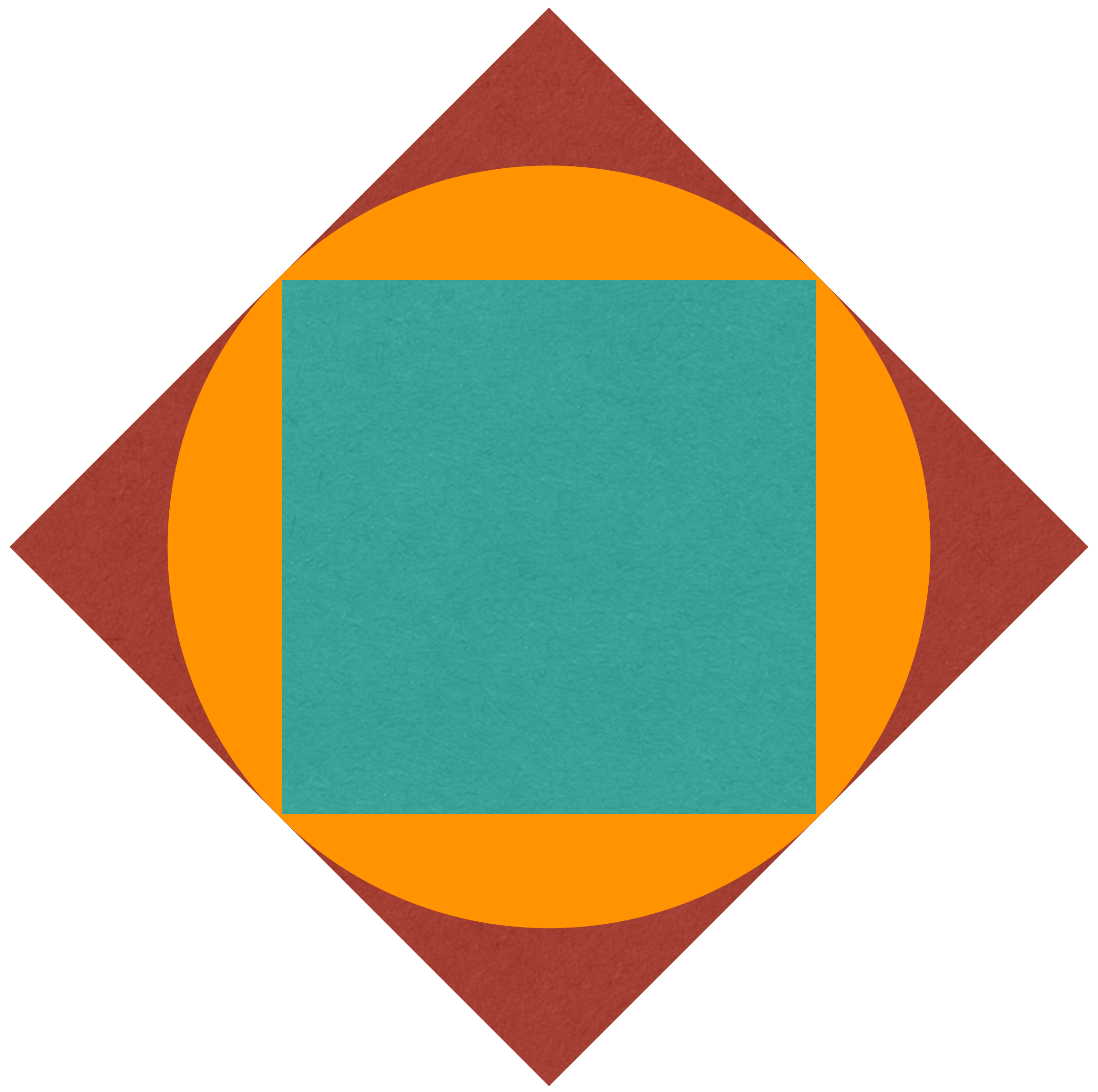

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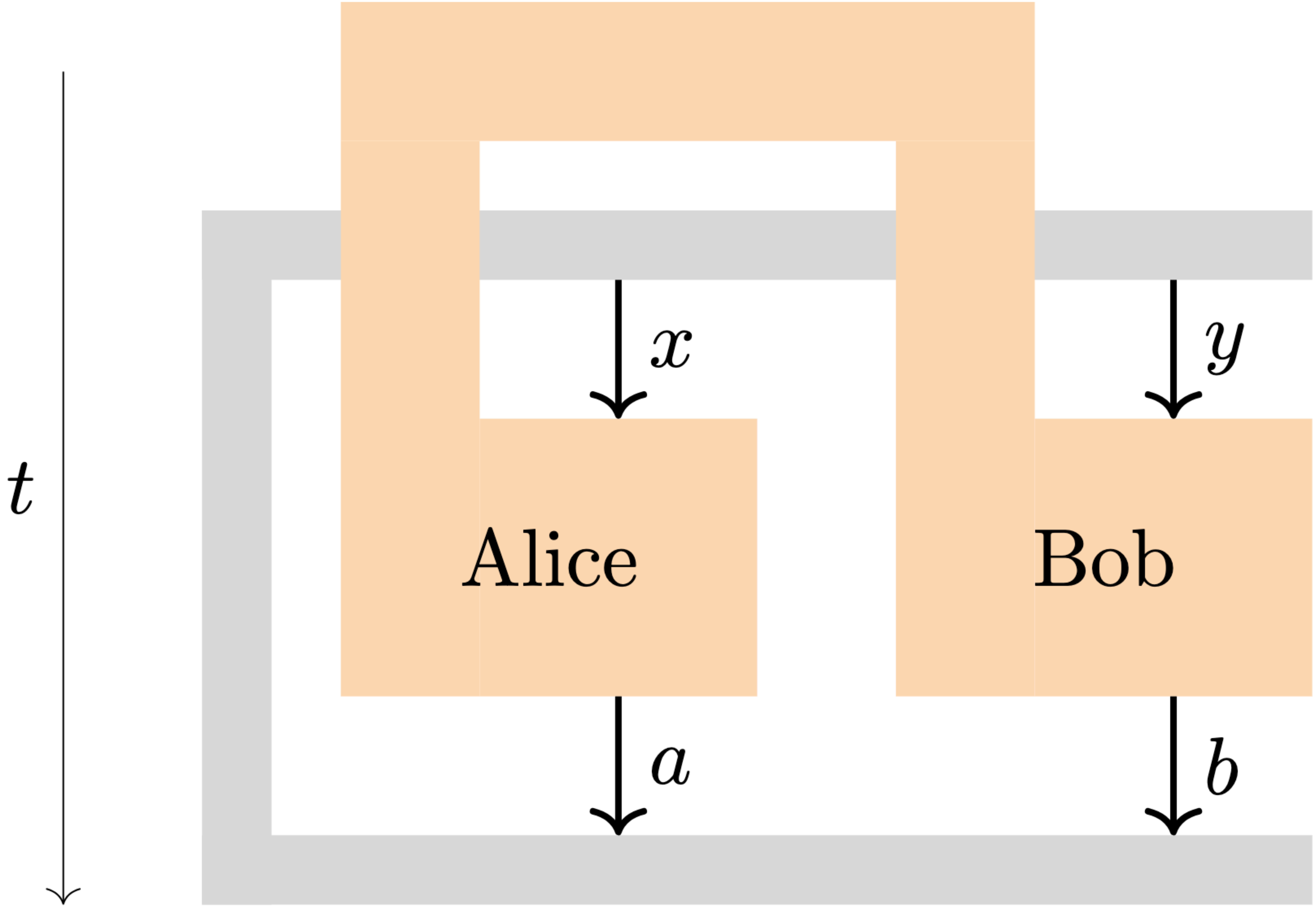

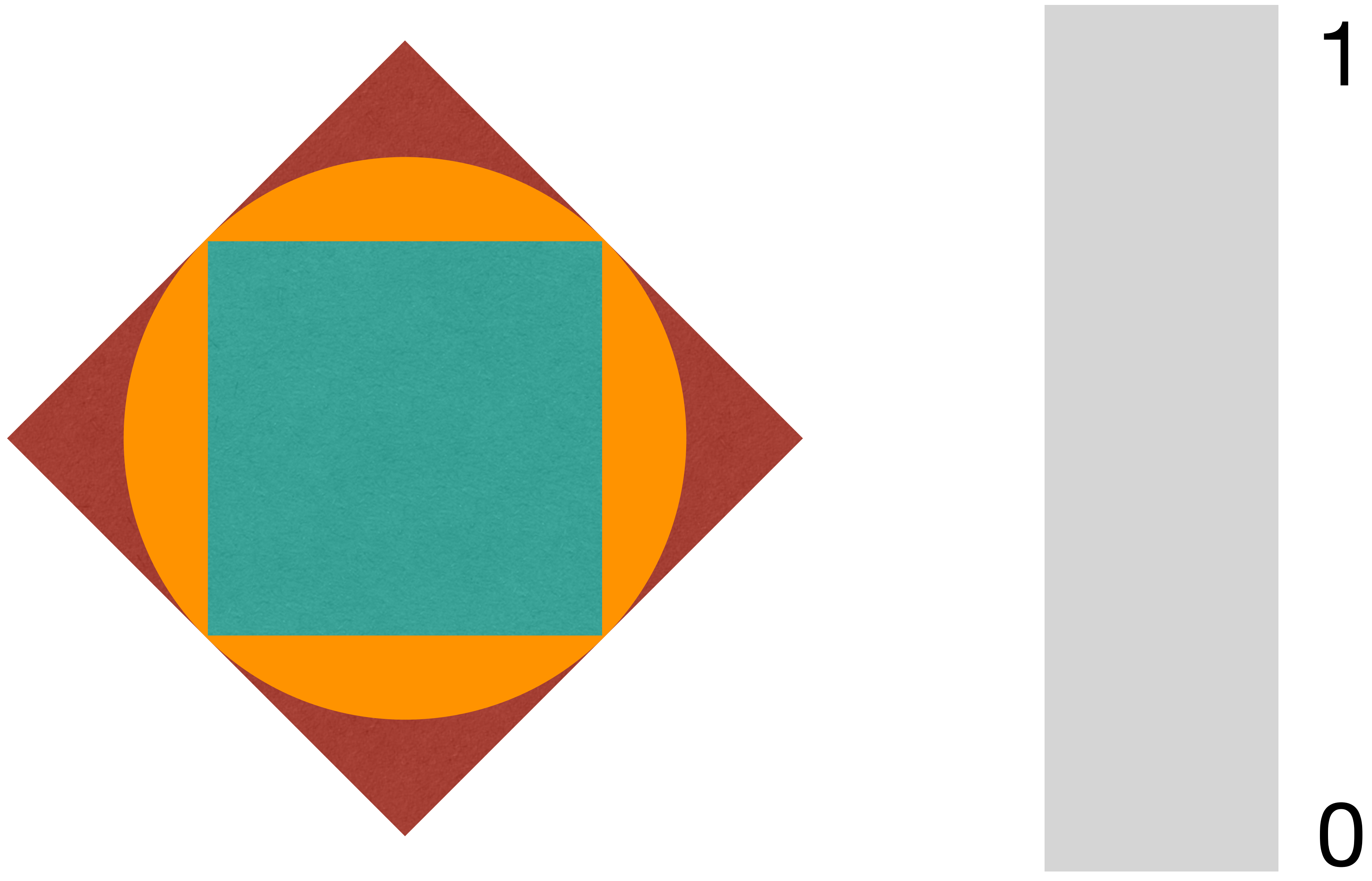

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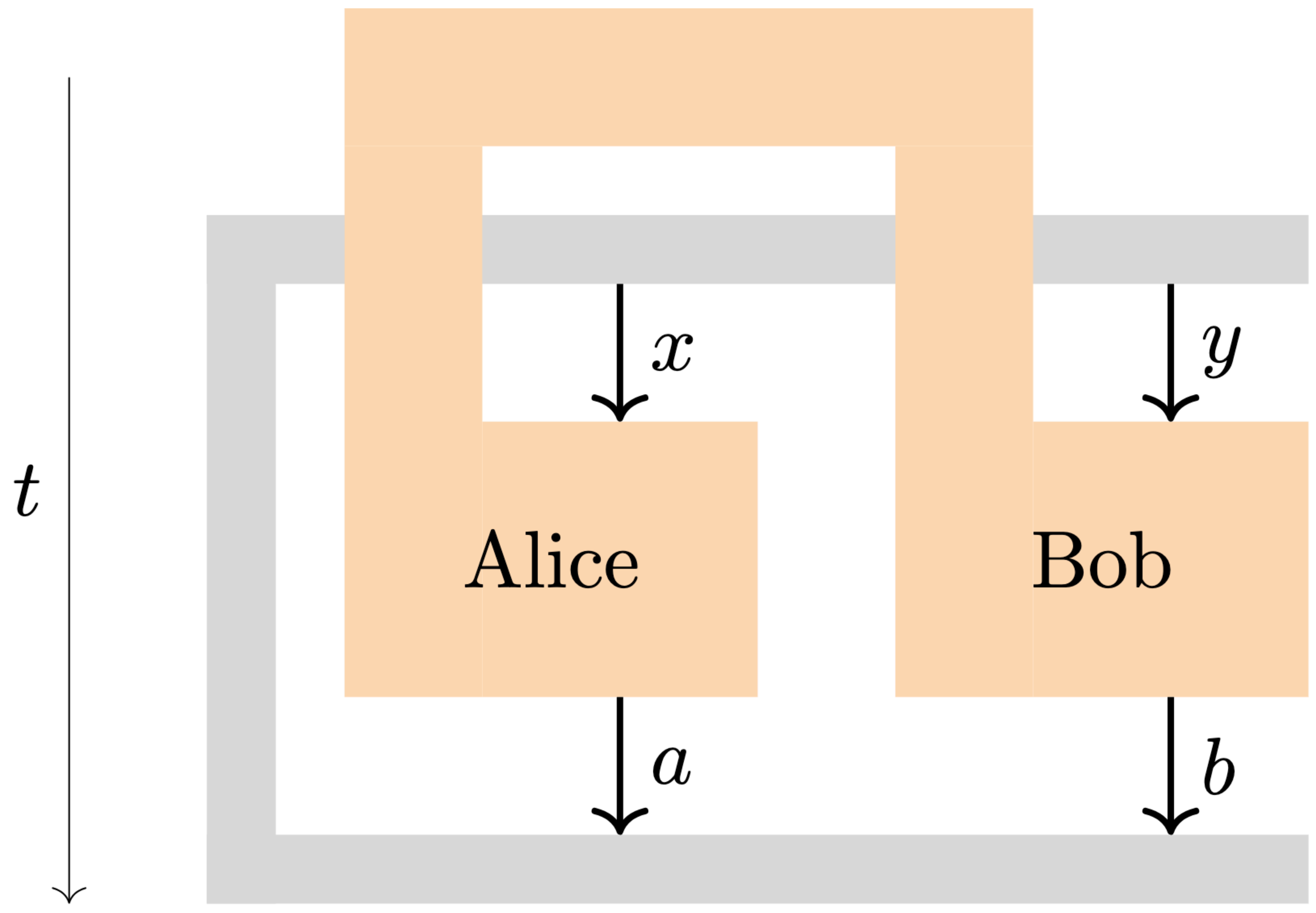

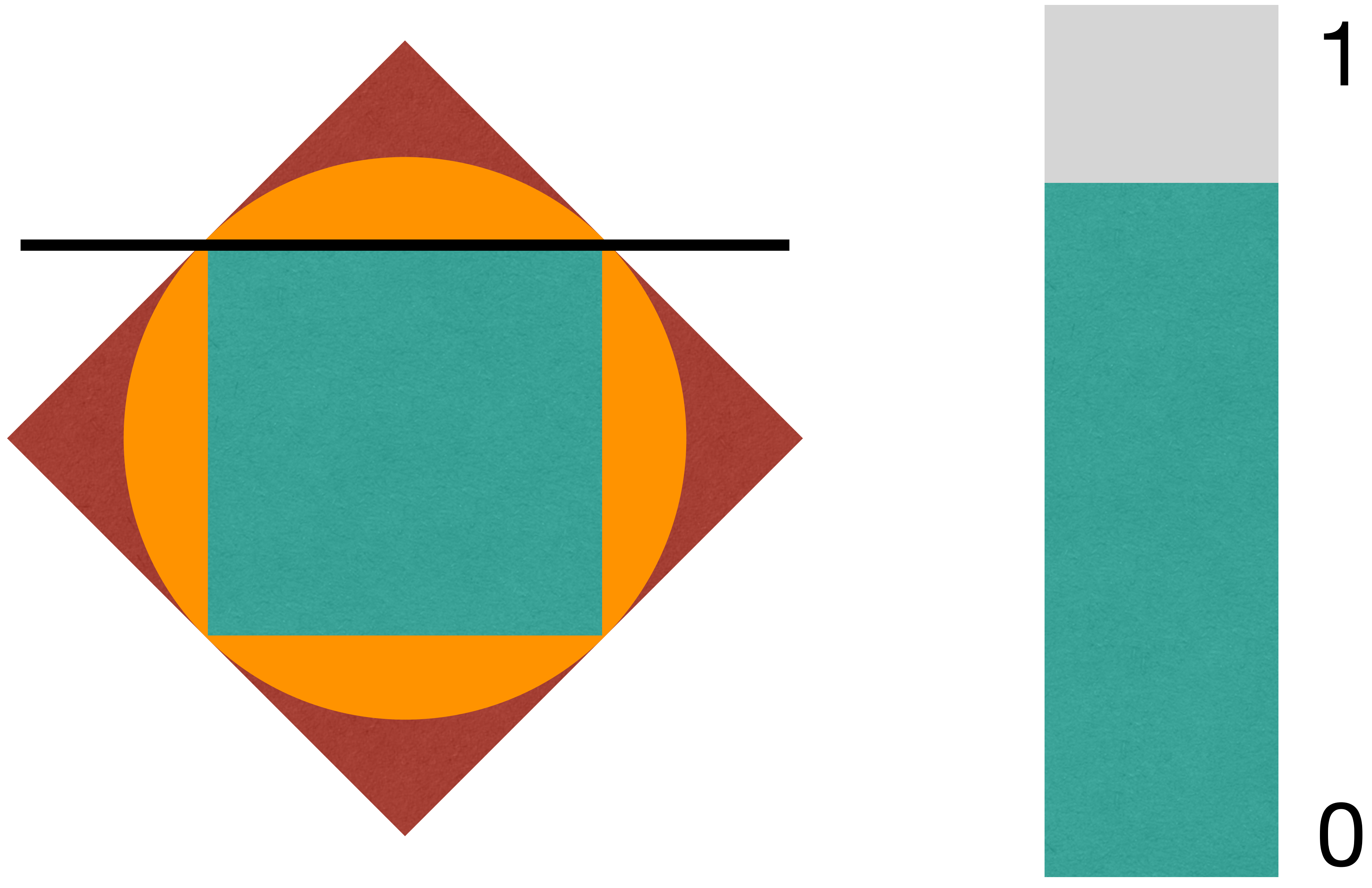

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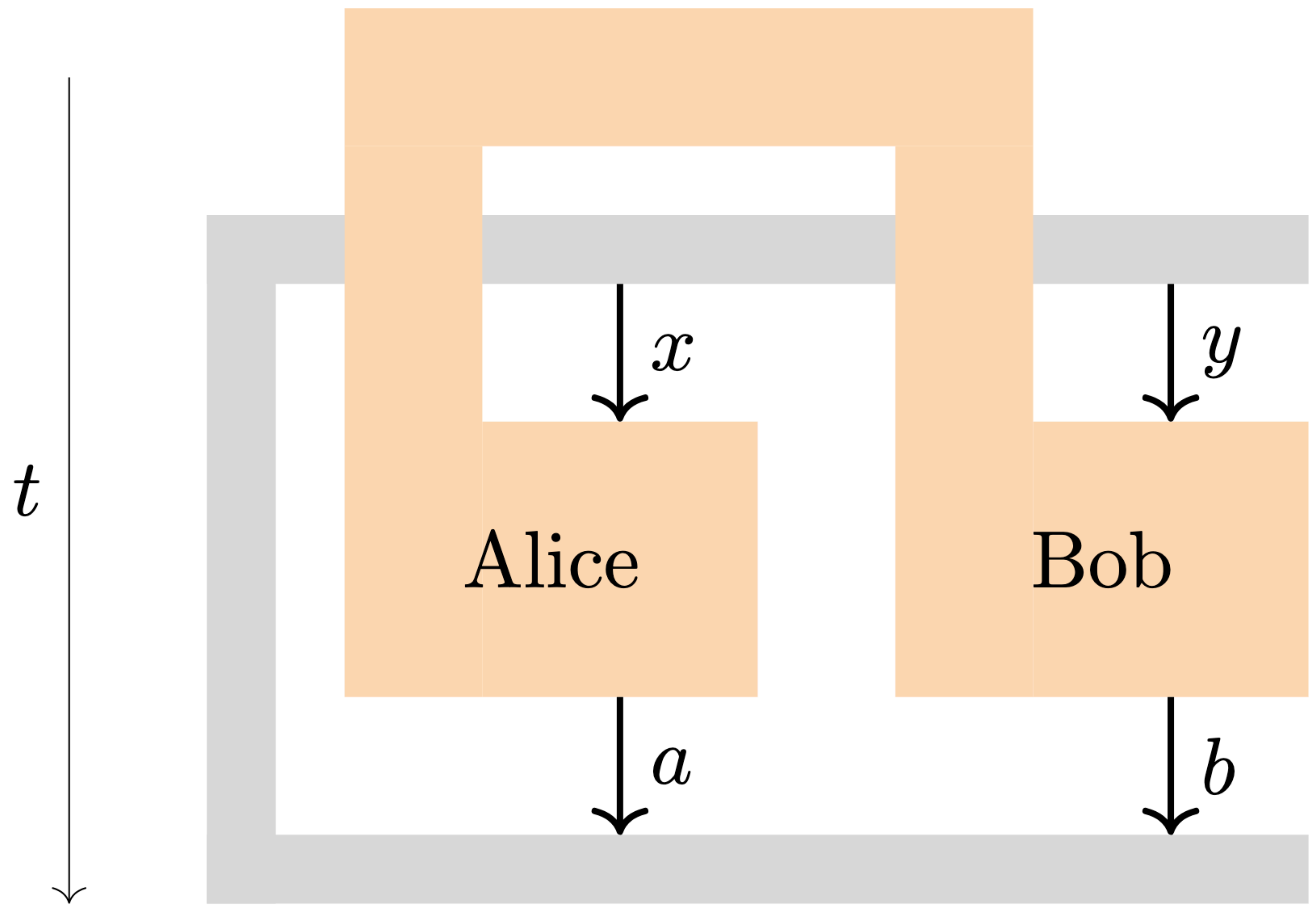

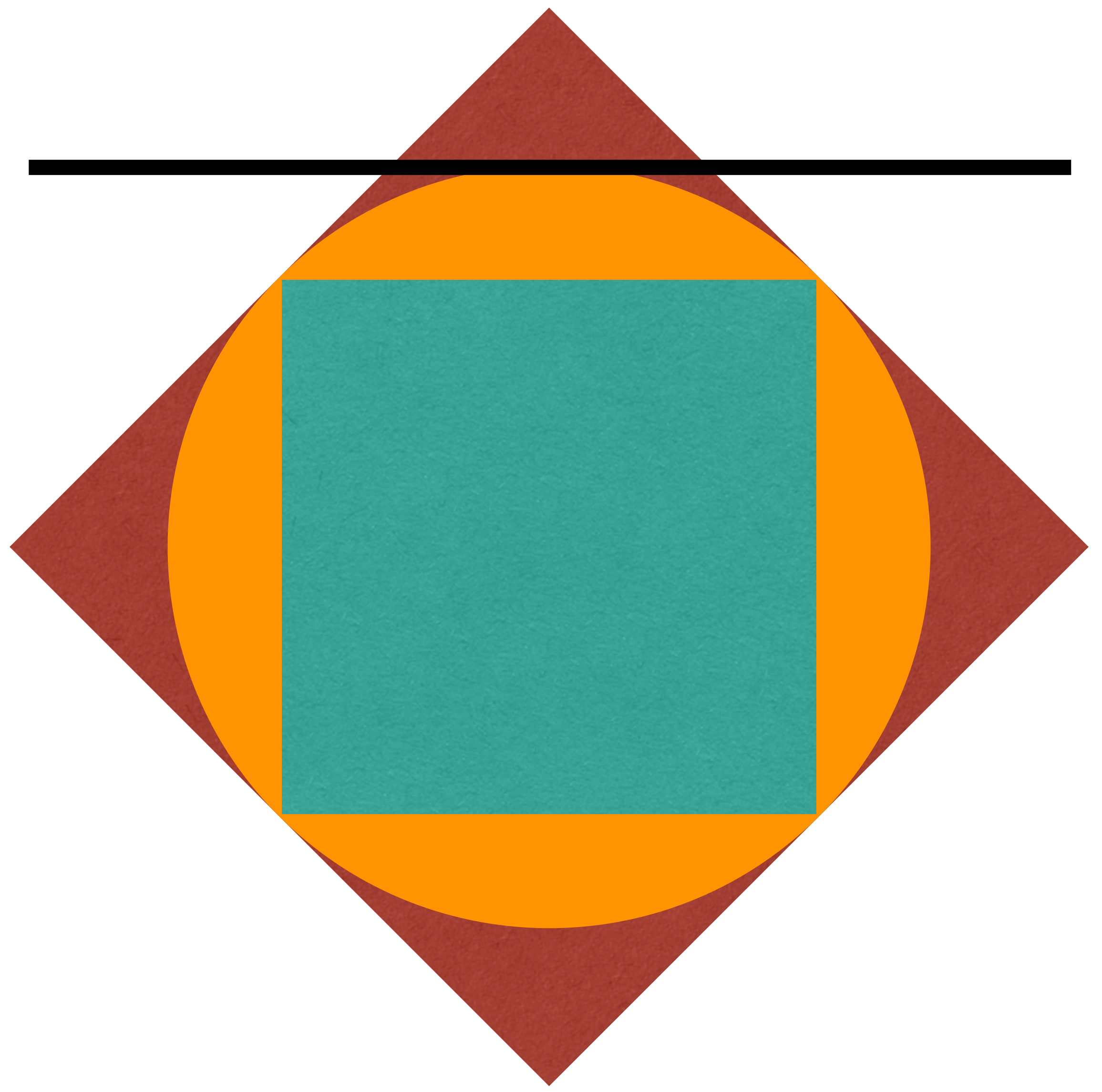

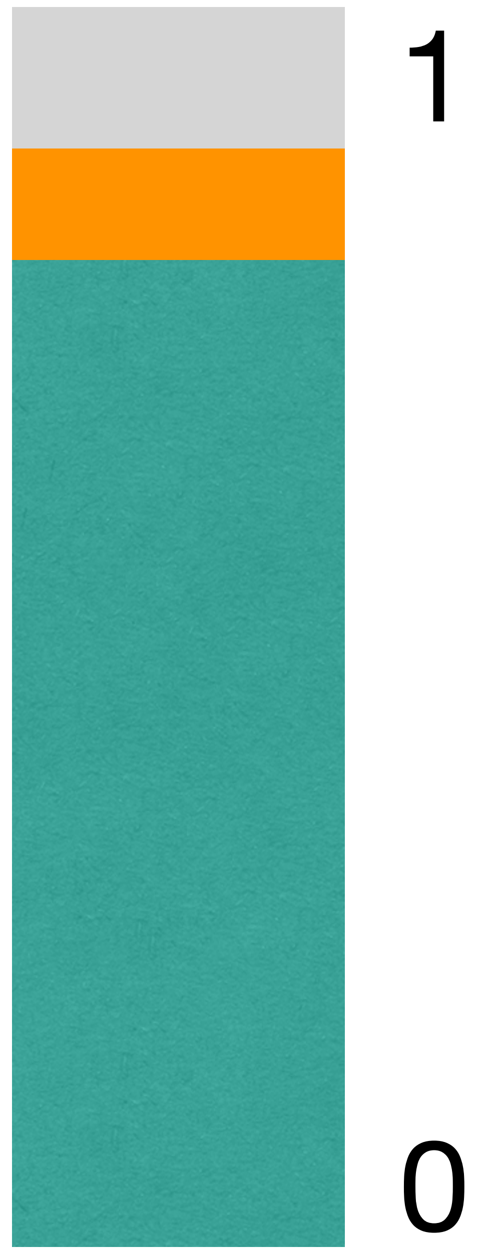

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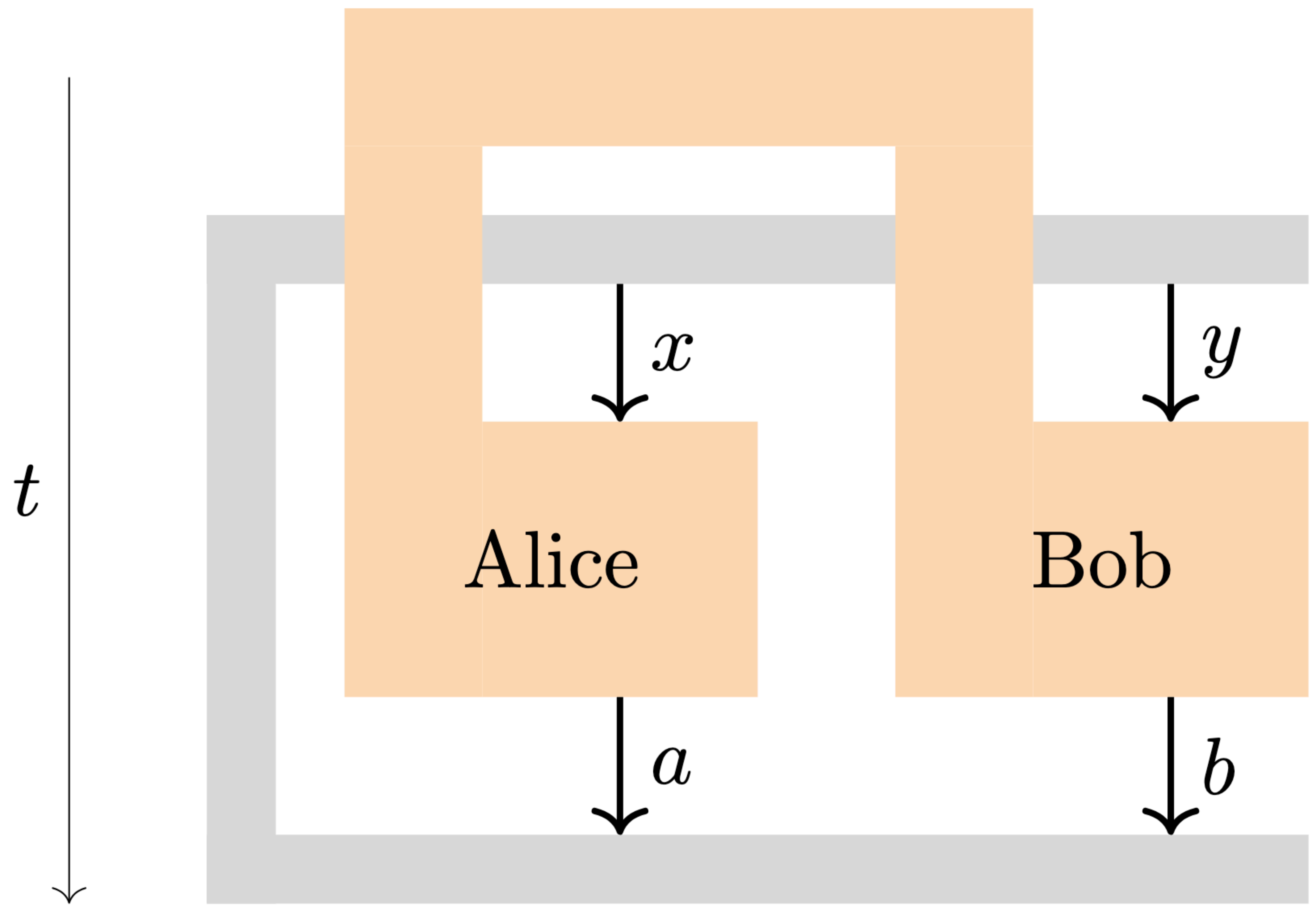
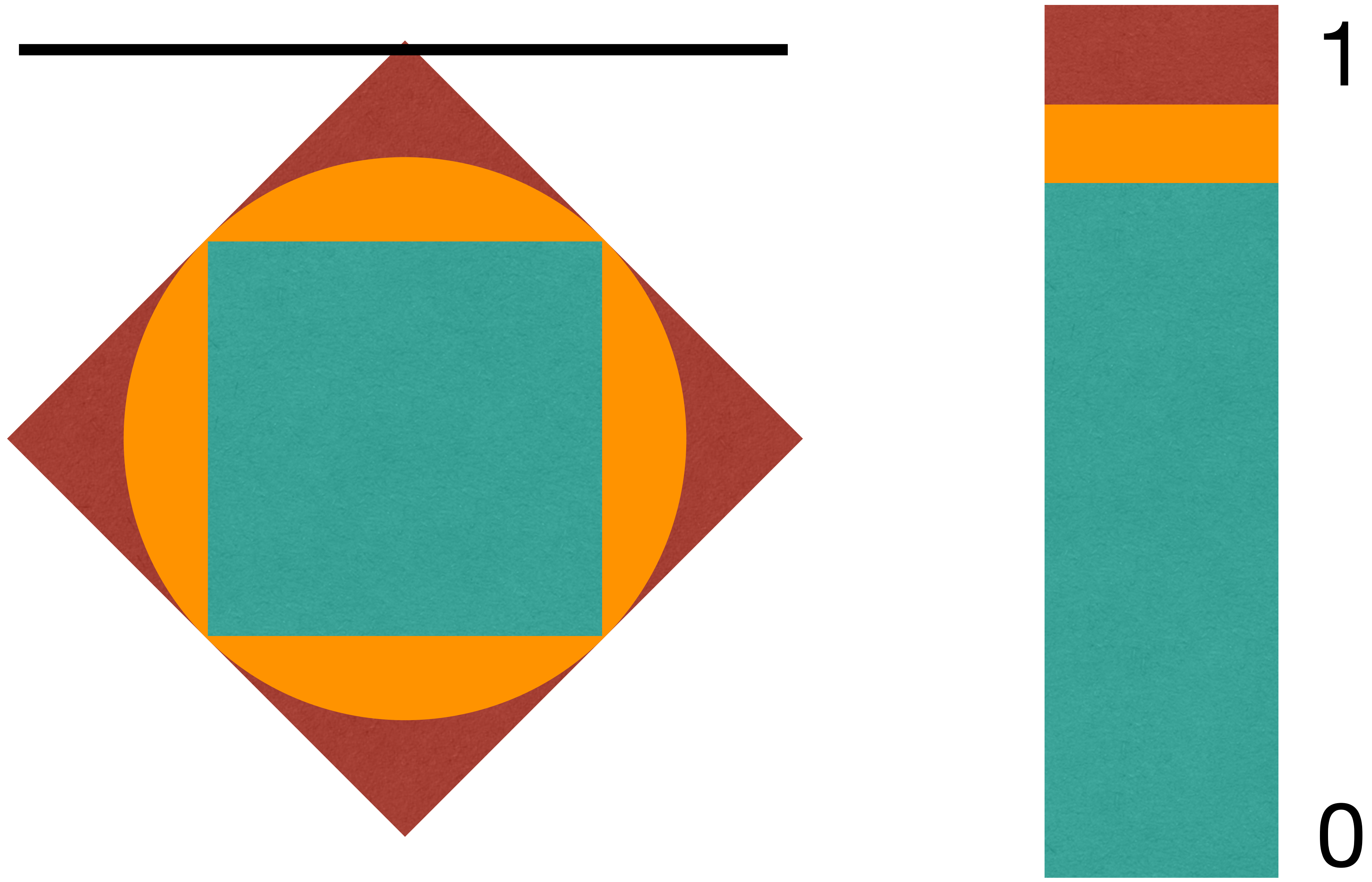
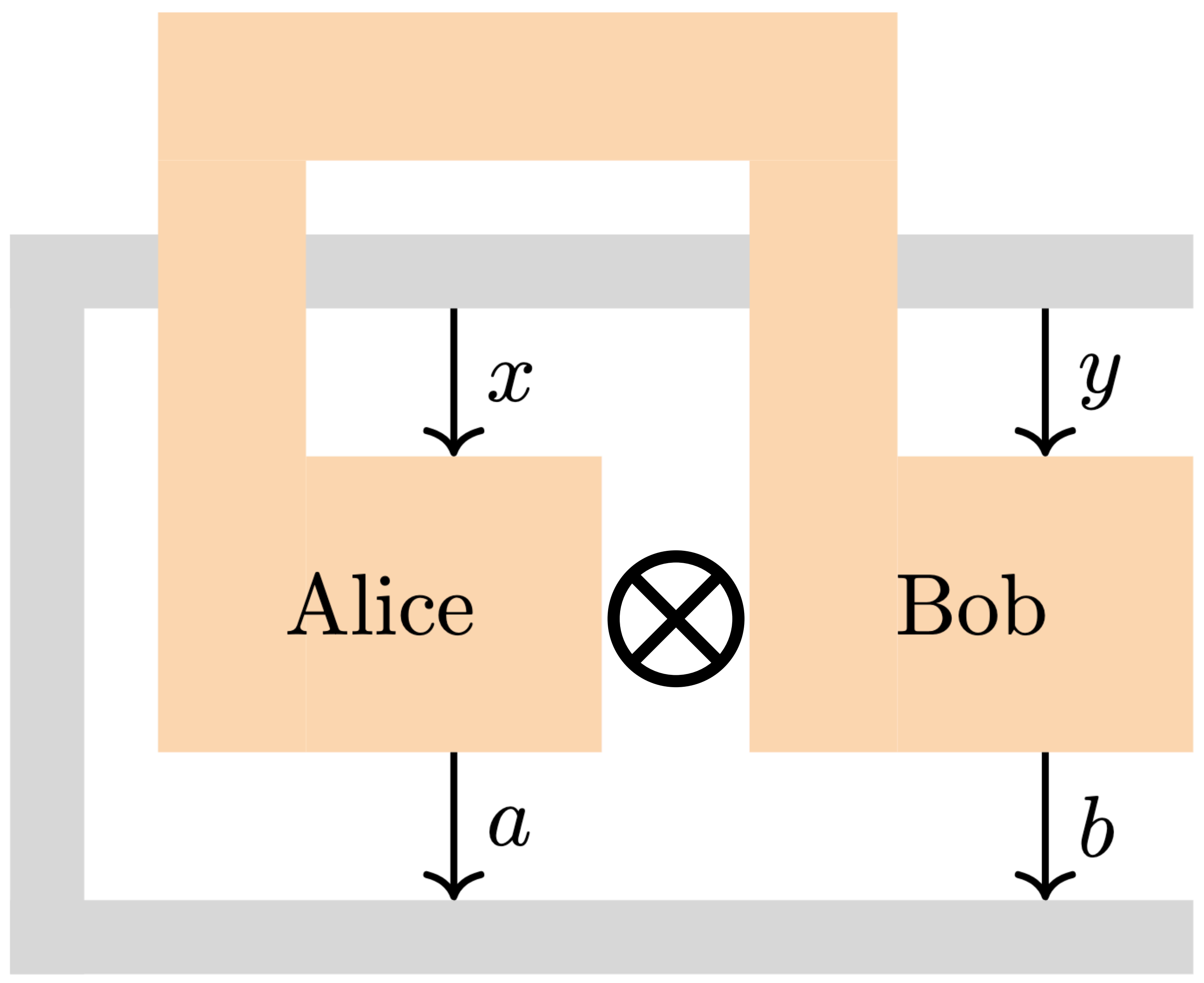
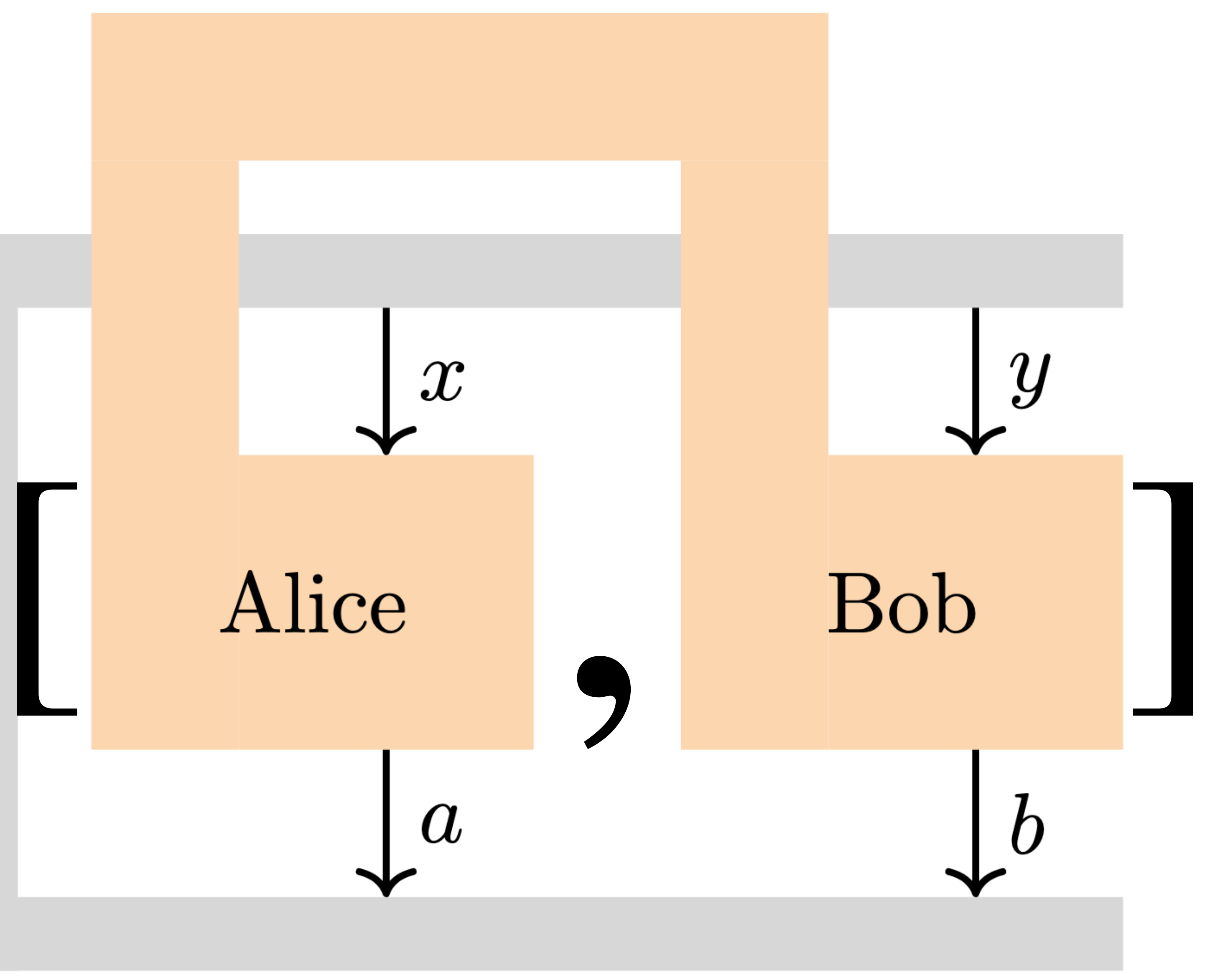




Non-locality 102

quantum vs. commuting operator





Non-locality 102

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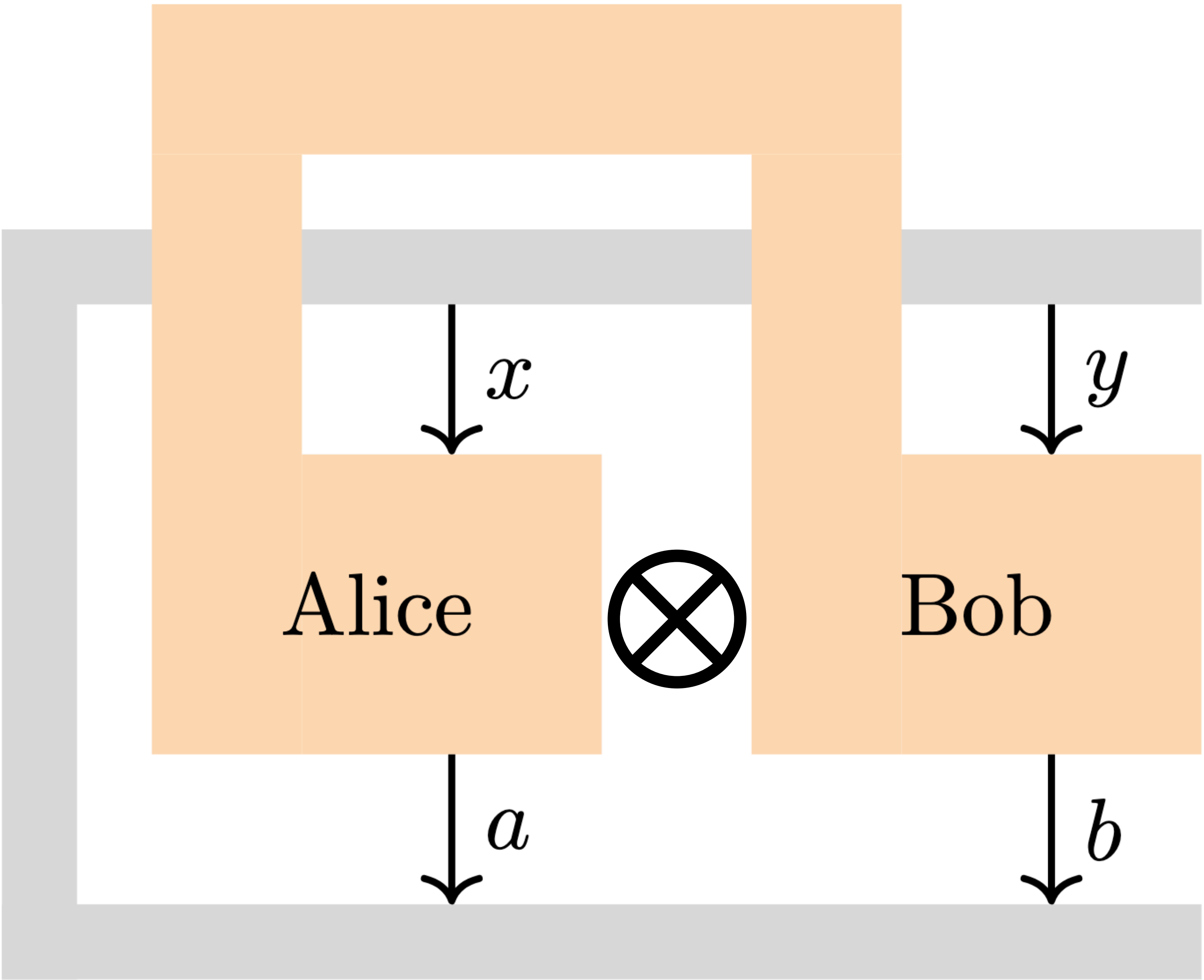
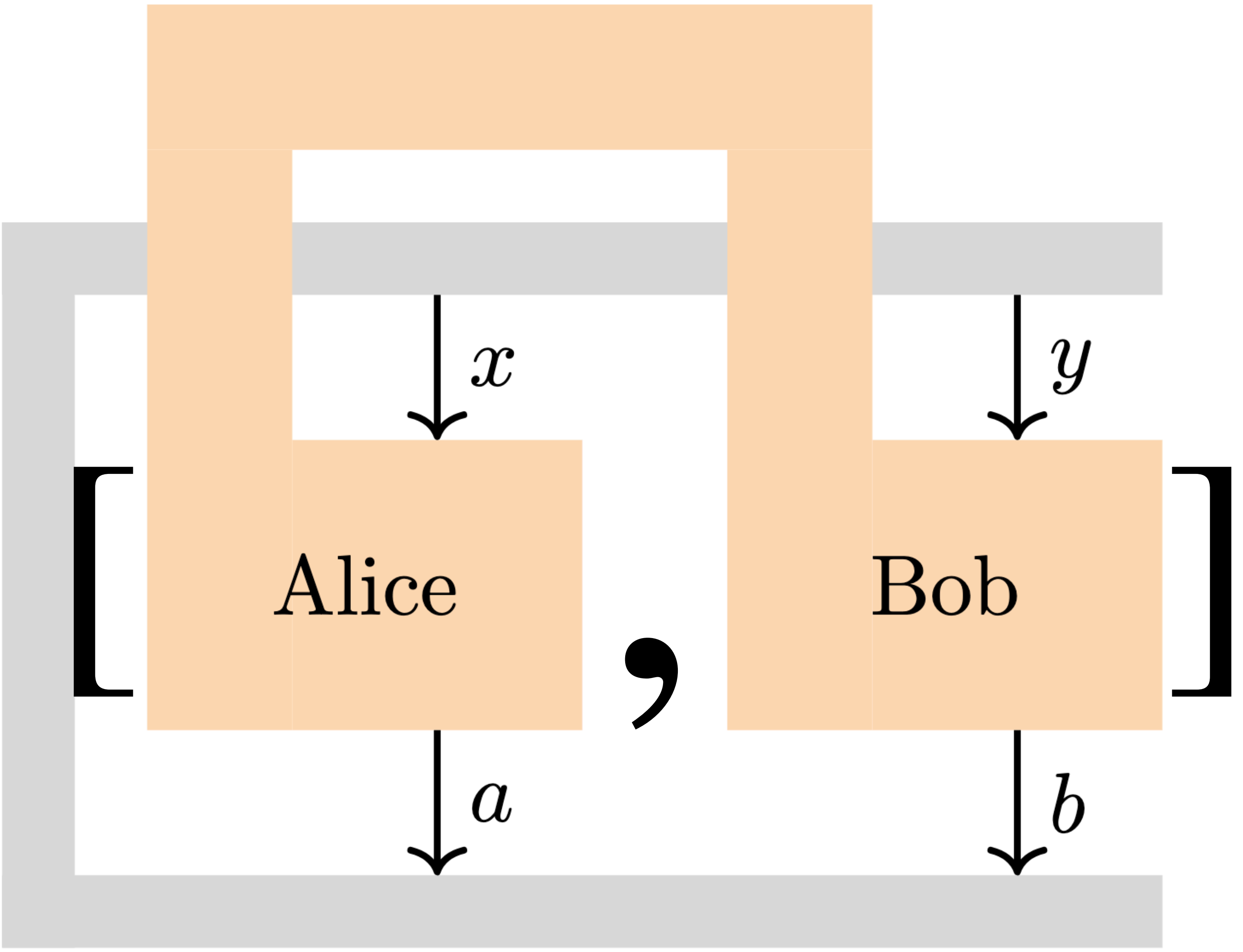


$\subsetneq$



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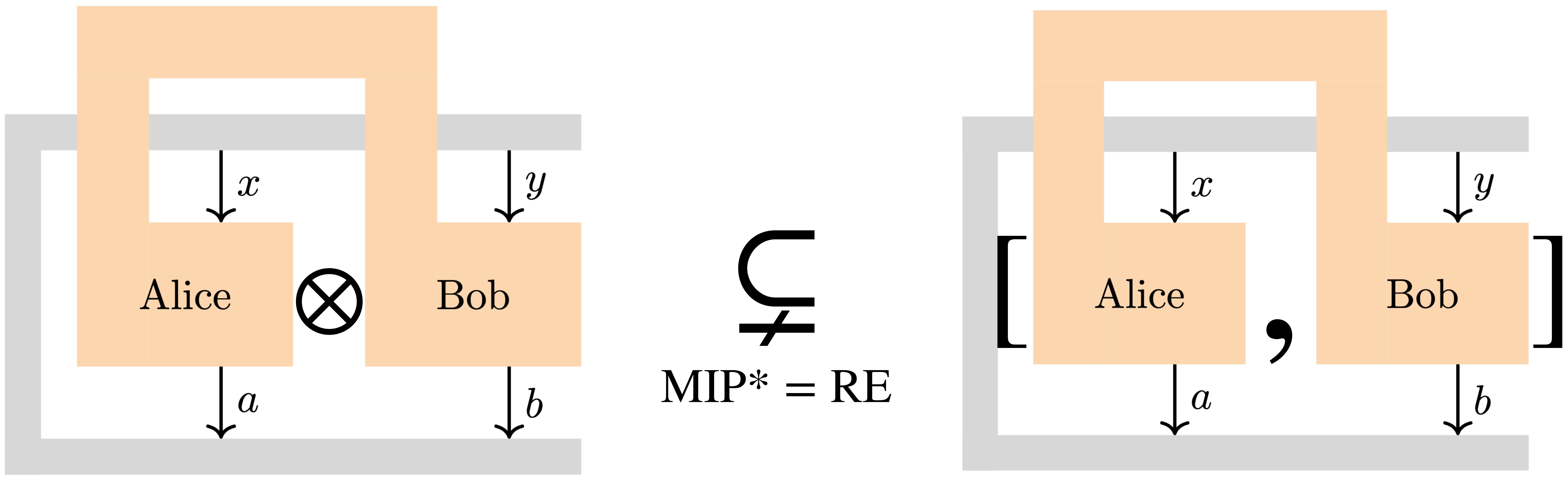

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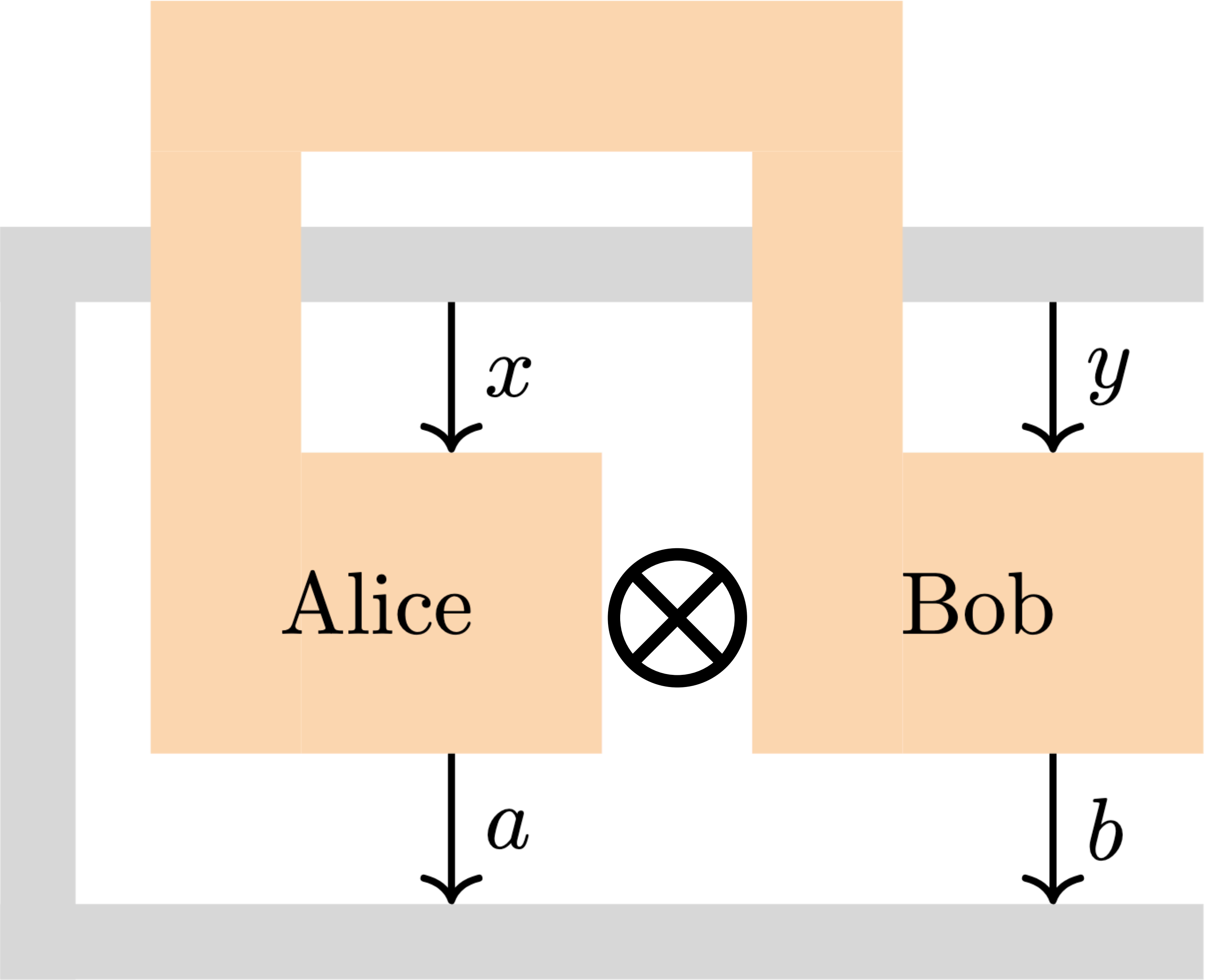
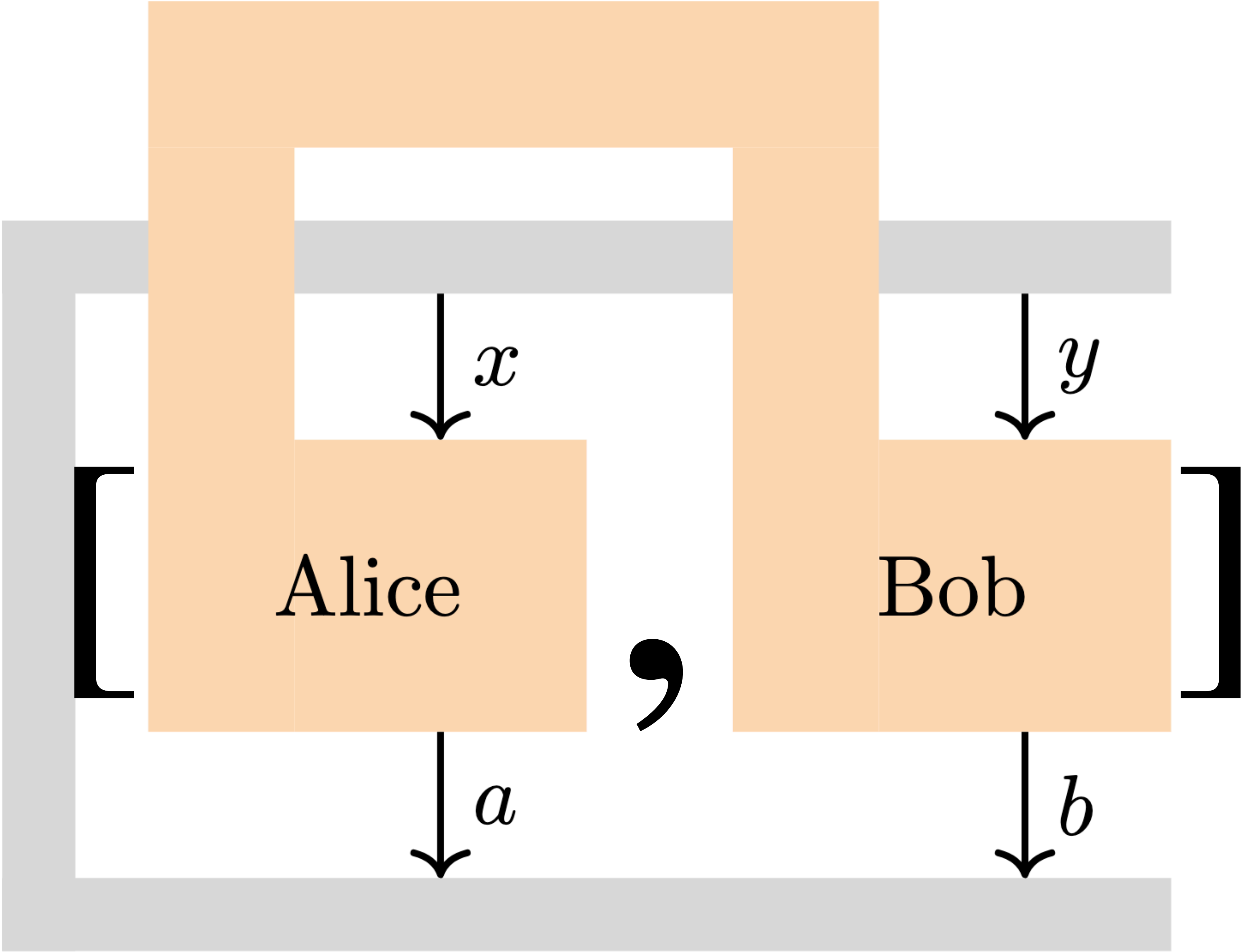


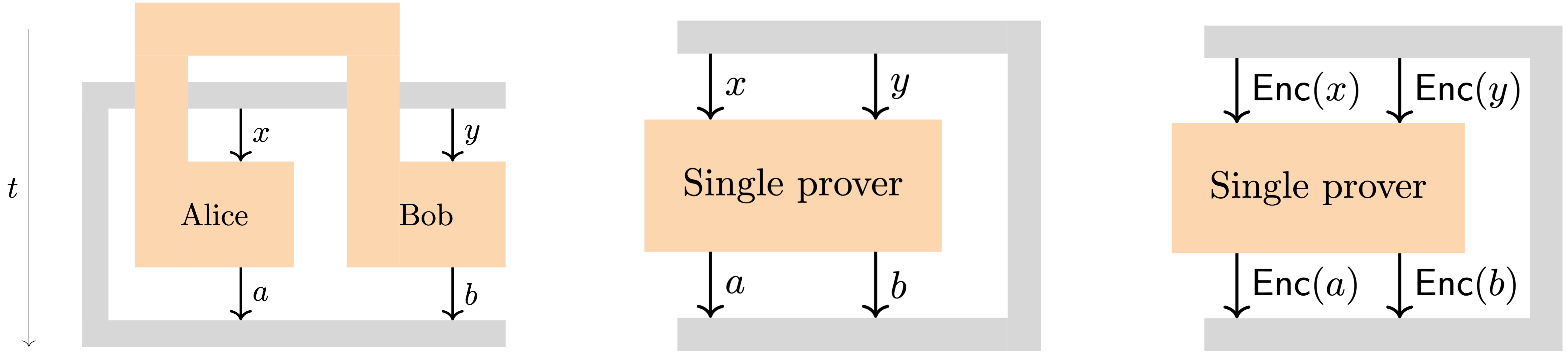
$$\subsetneq$$

$\text{MIP}^* = \text{RE}$

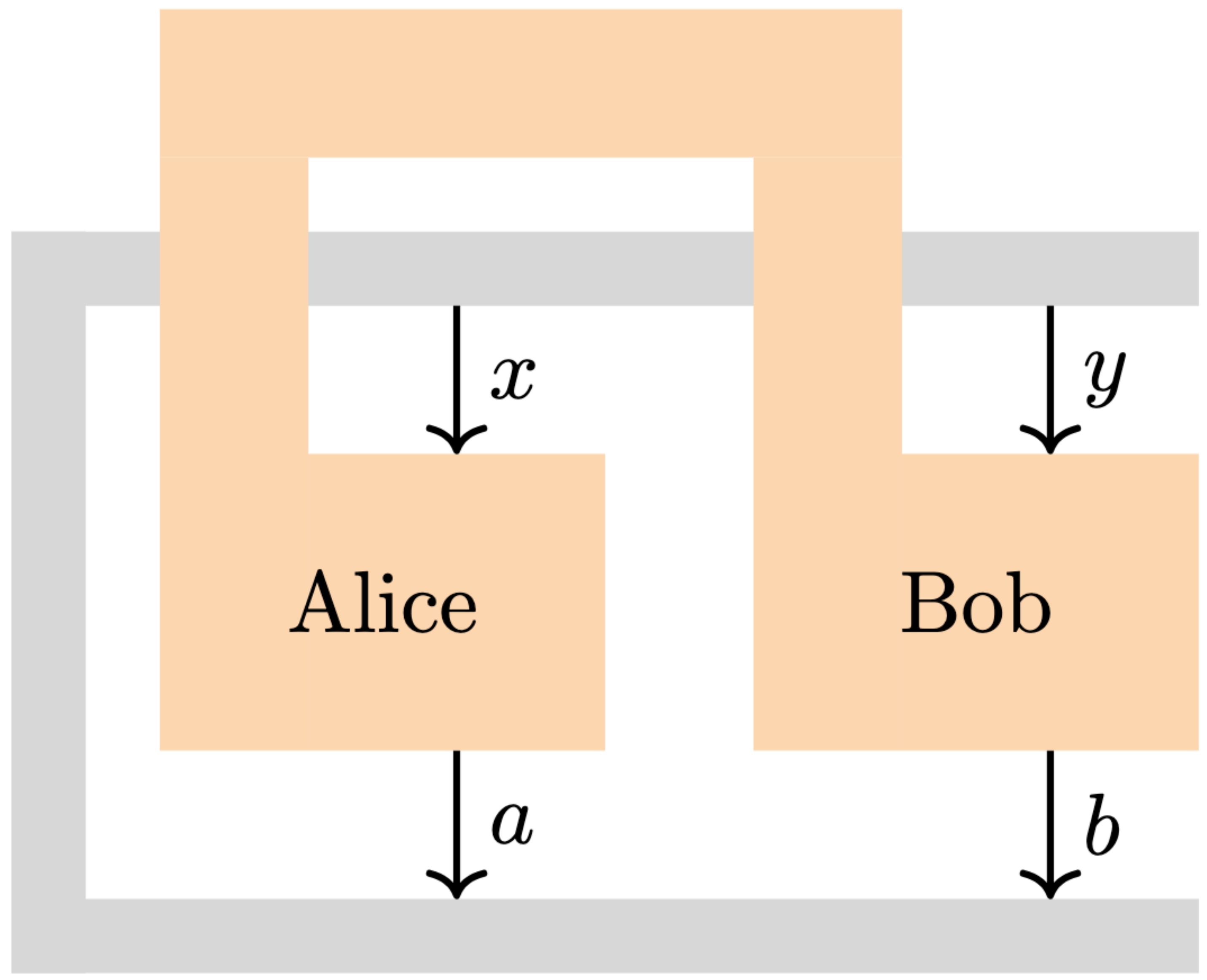
$= \text{if dim } < \infty$

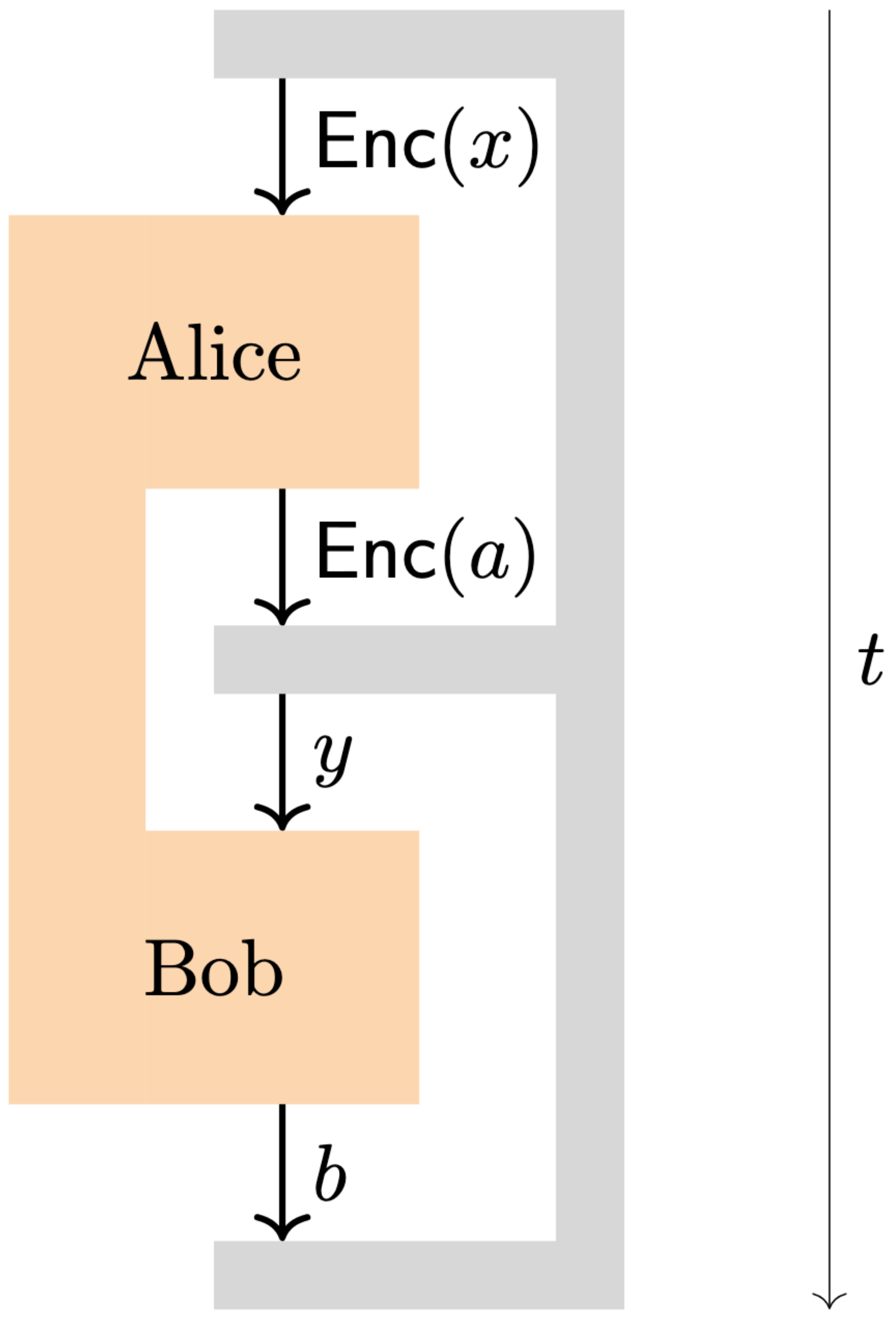


Removing space-like separation using cryptography



KLVY compiler





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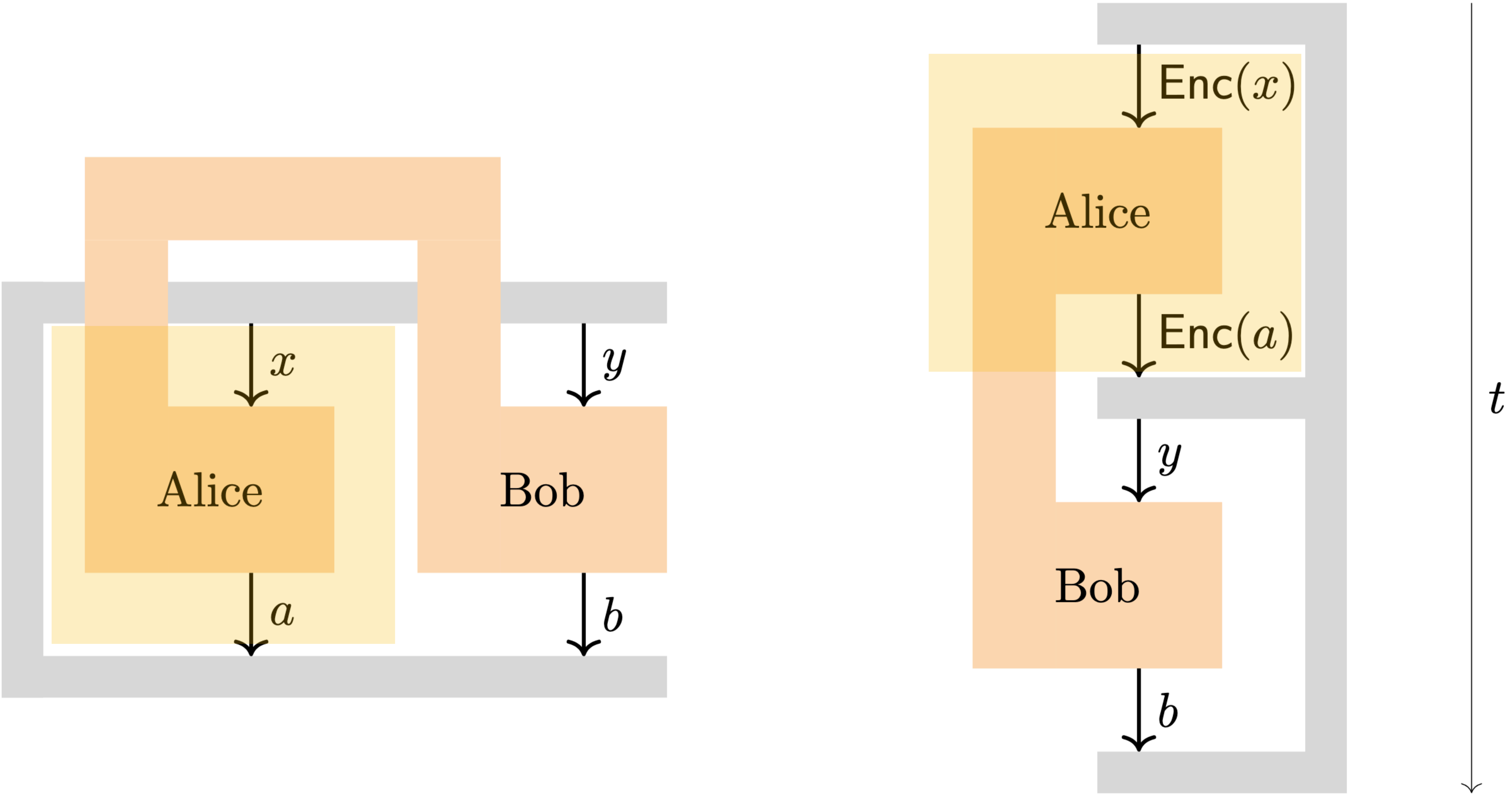

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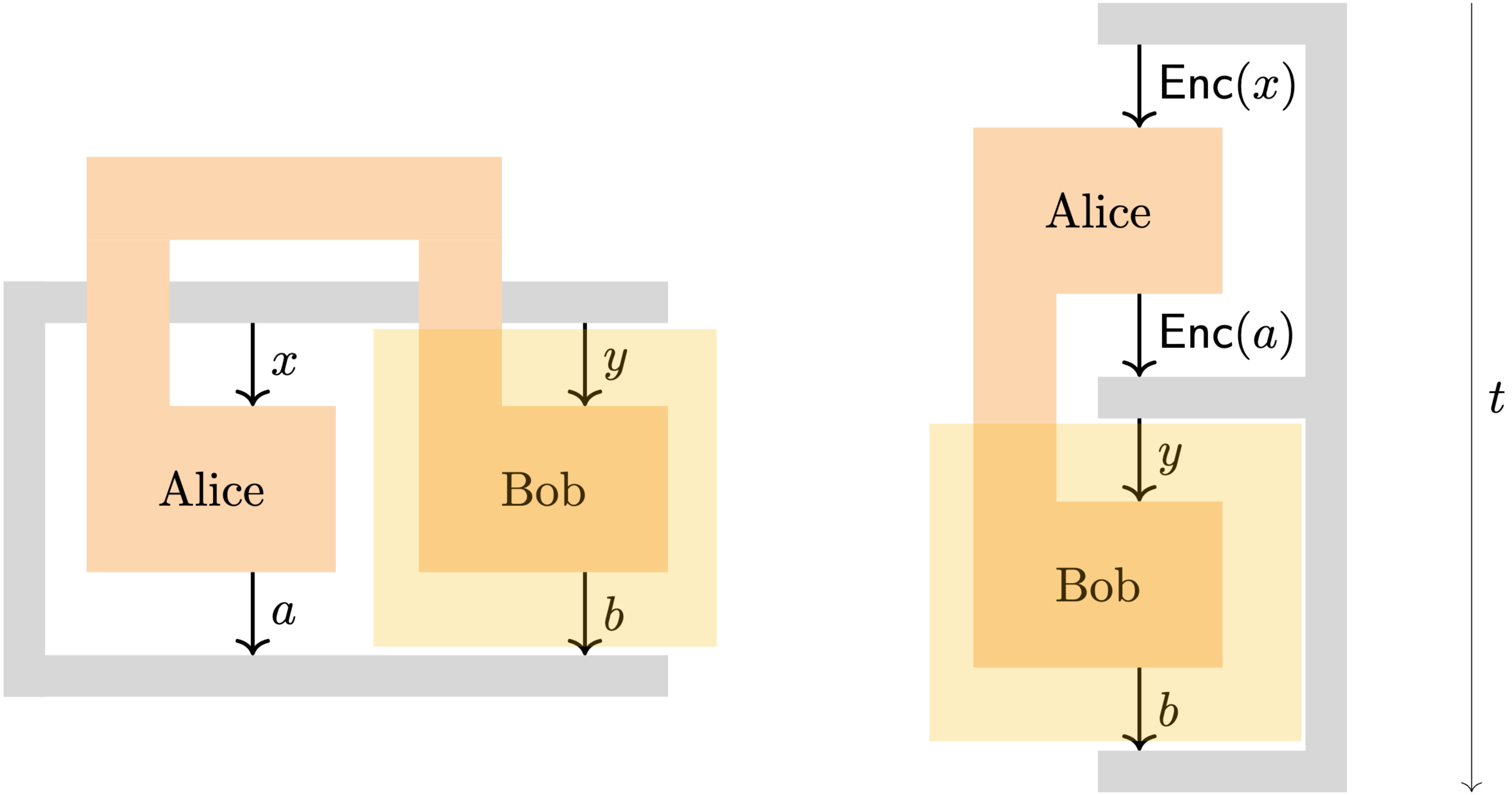

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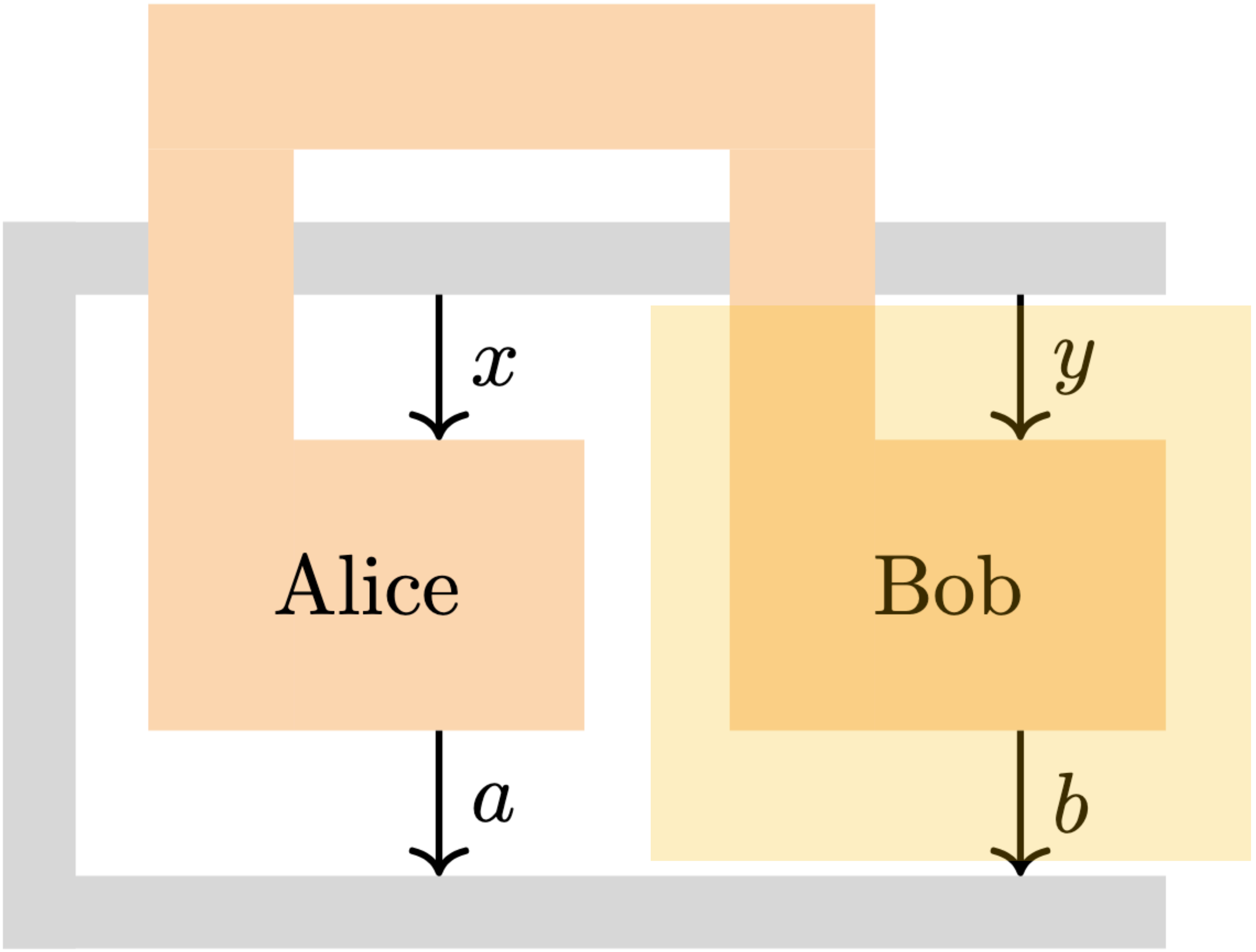
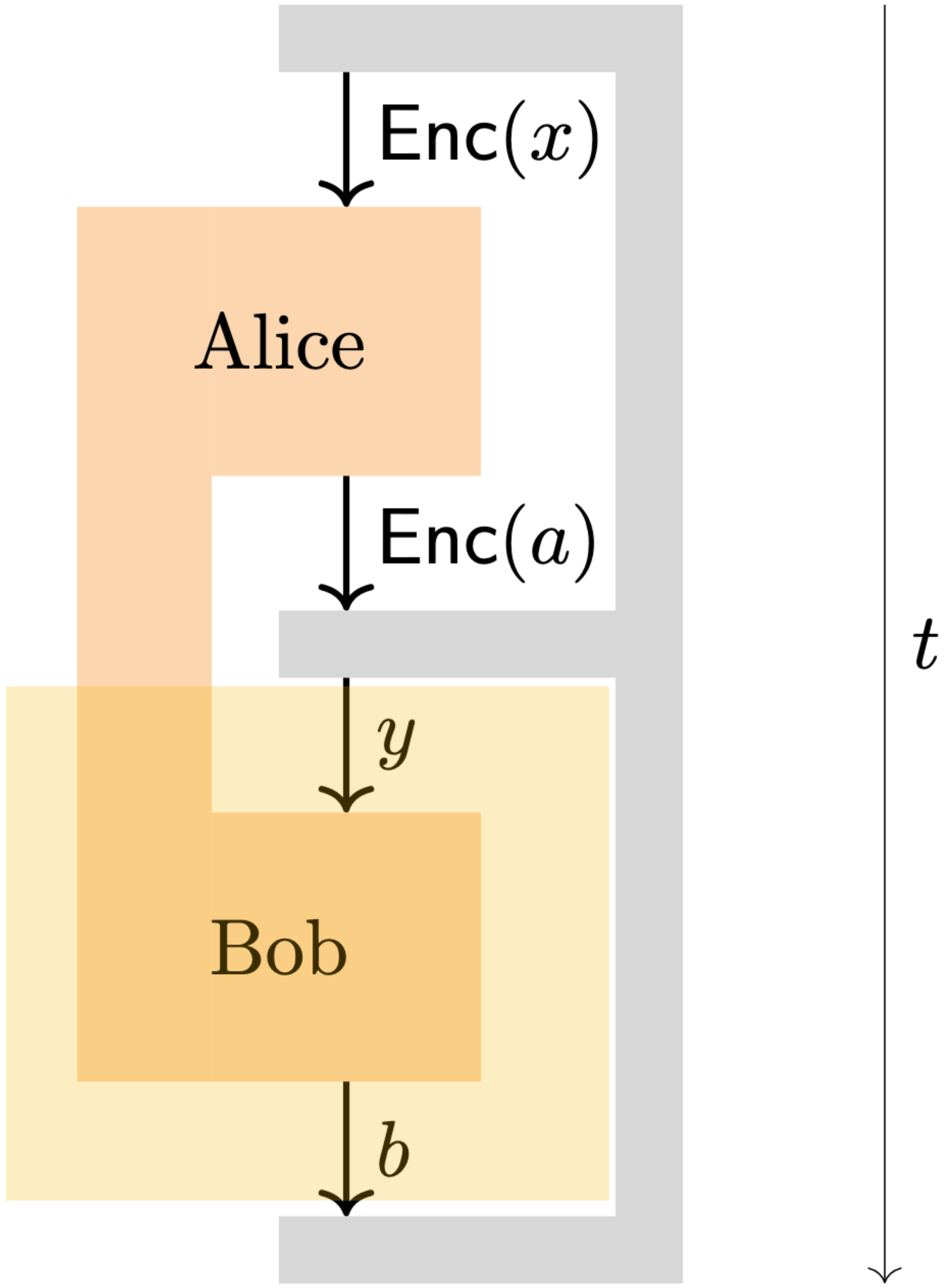
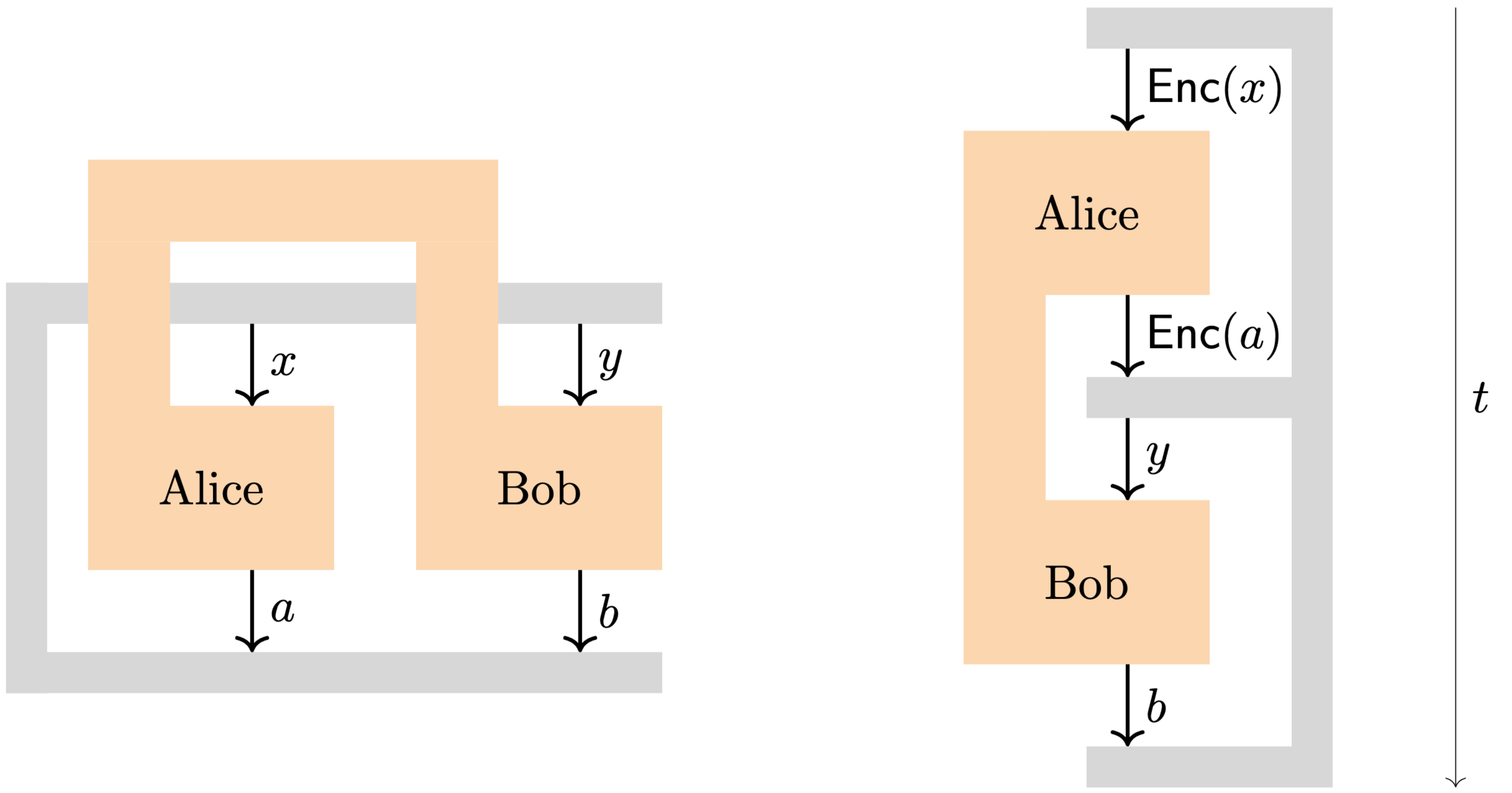




Works for all k players games !

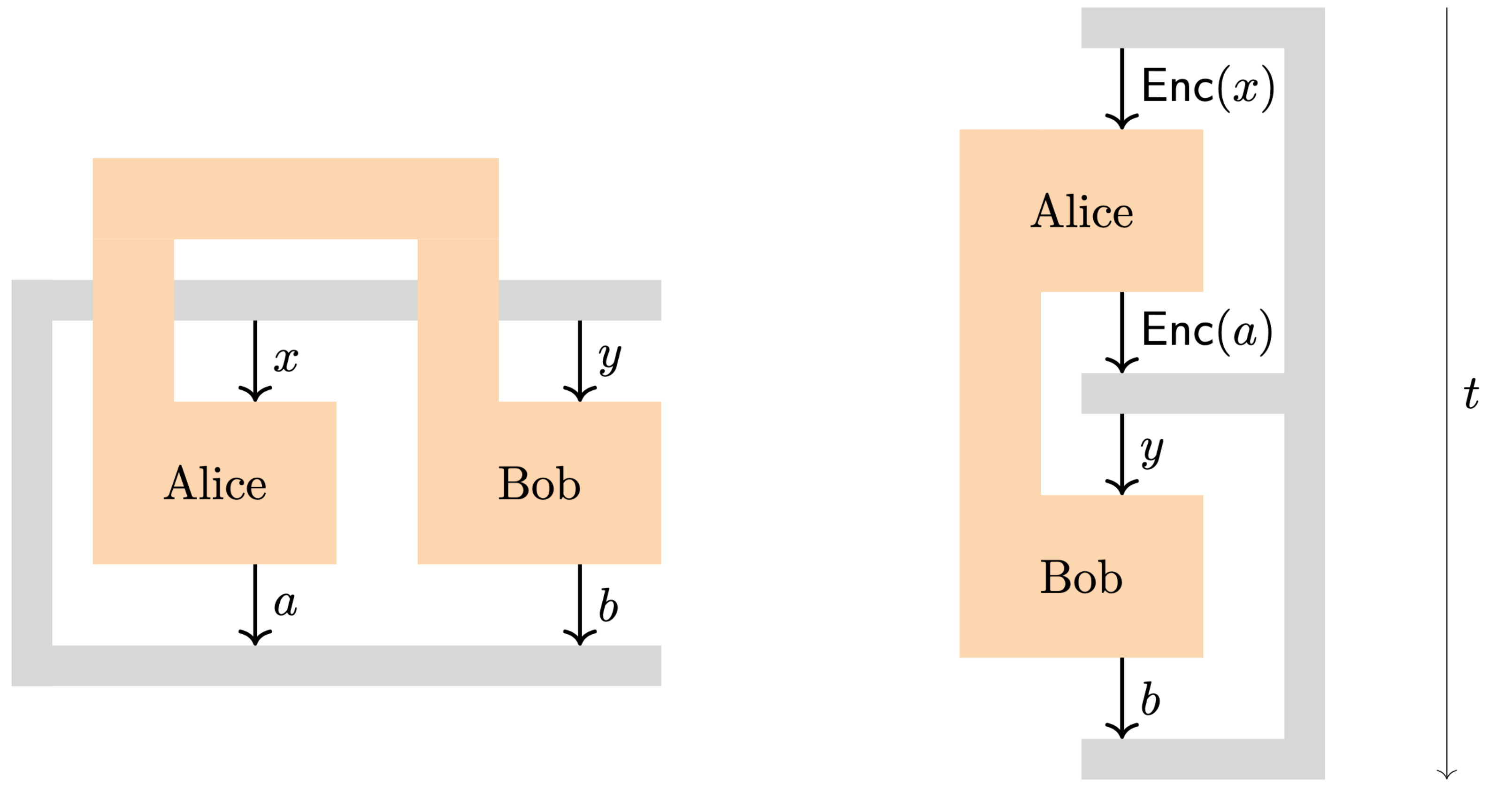
KLVY compiler : QFHE



Tool: Quantum Fully Homomorphic Encryption scheme (QFHE) with

- correctness with auxiliary input
- IND-CPA security against quantum polynomial-time (QPT) adversaries

KLVY compiler : QFHE



λ security parameter

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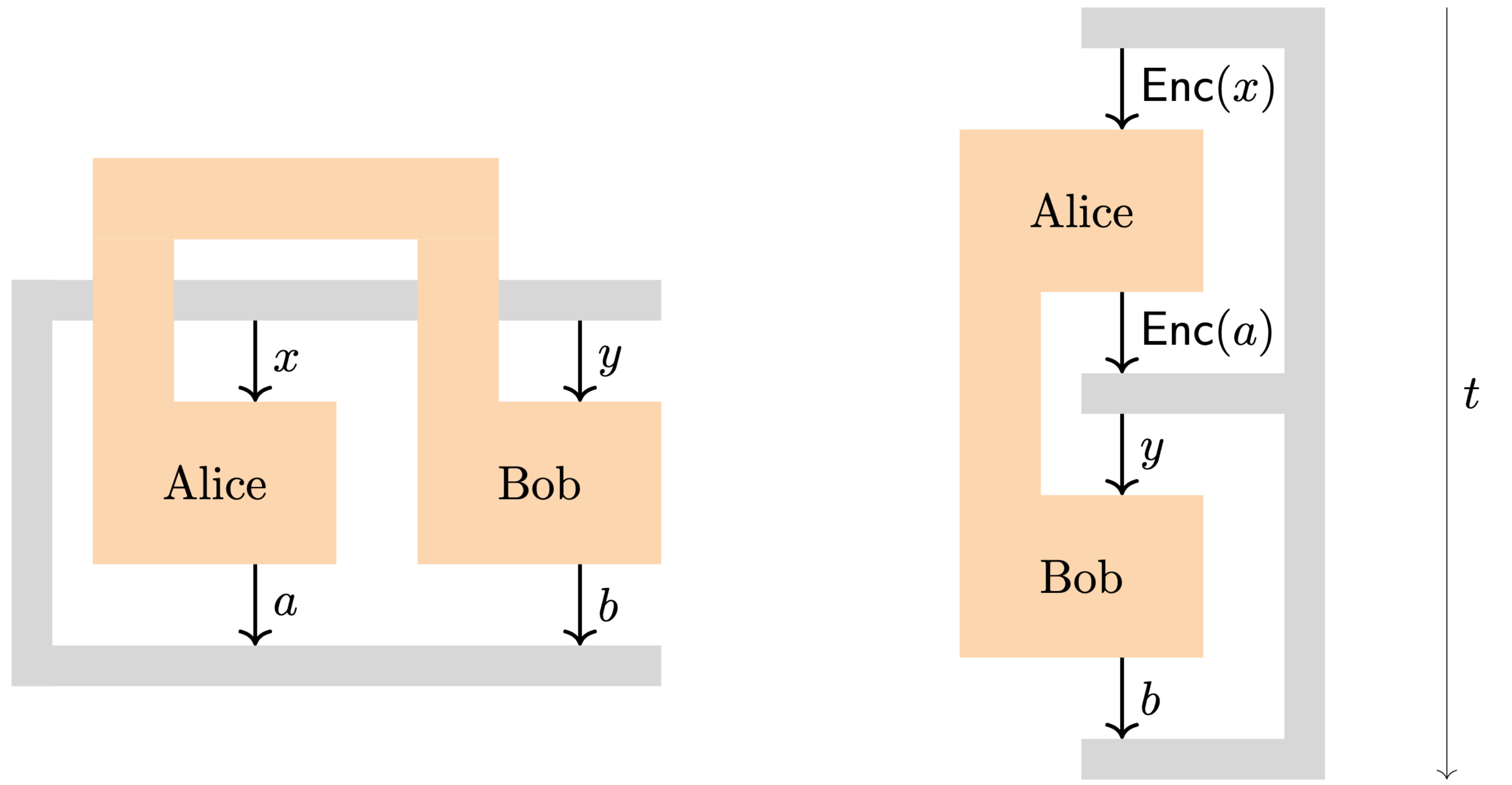

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Efficient provers: circuits poly(λ) gates

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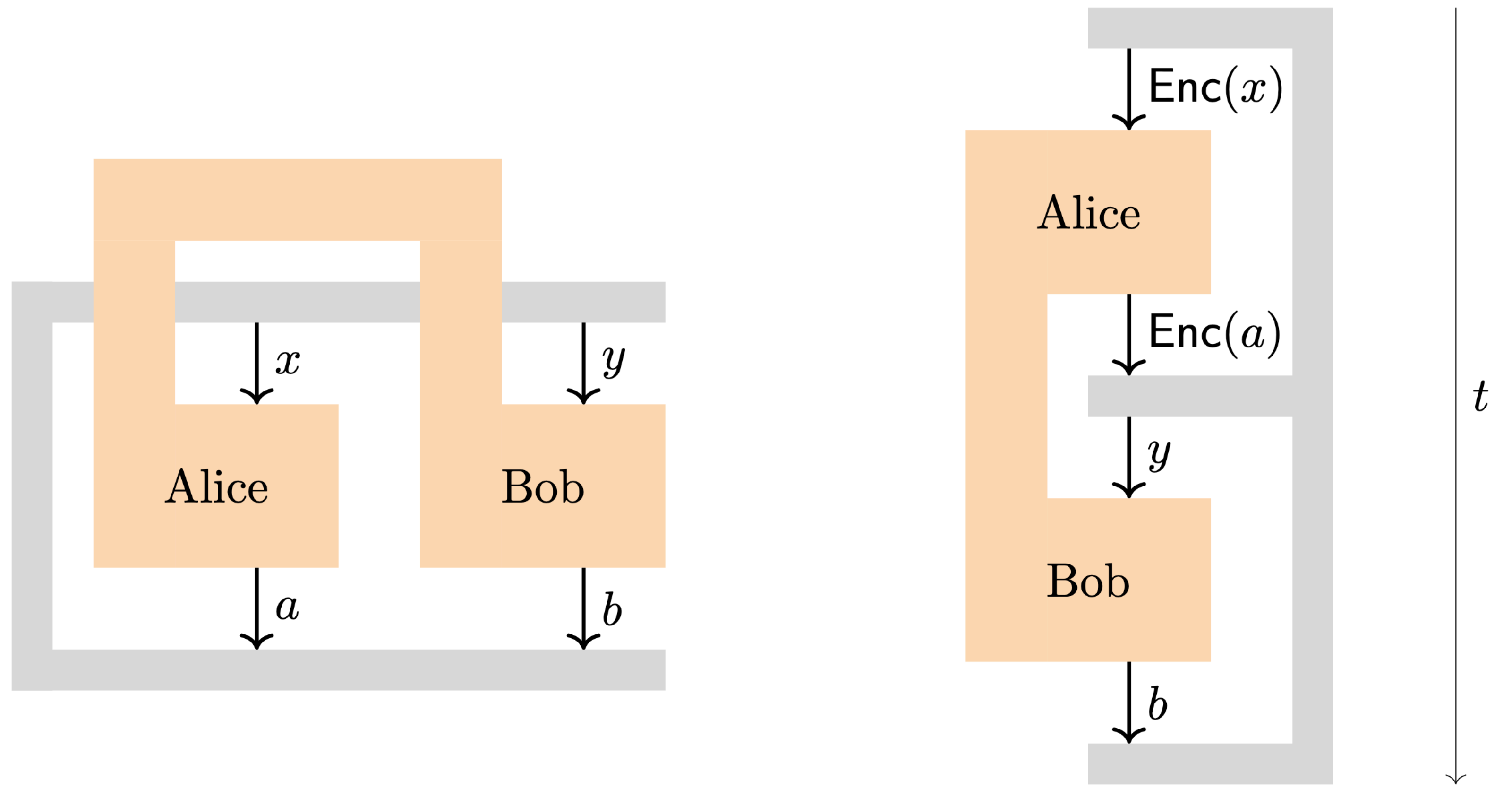

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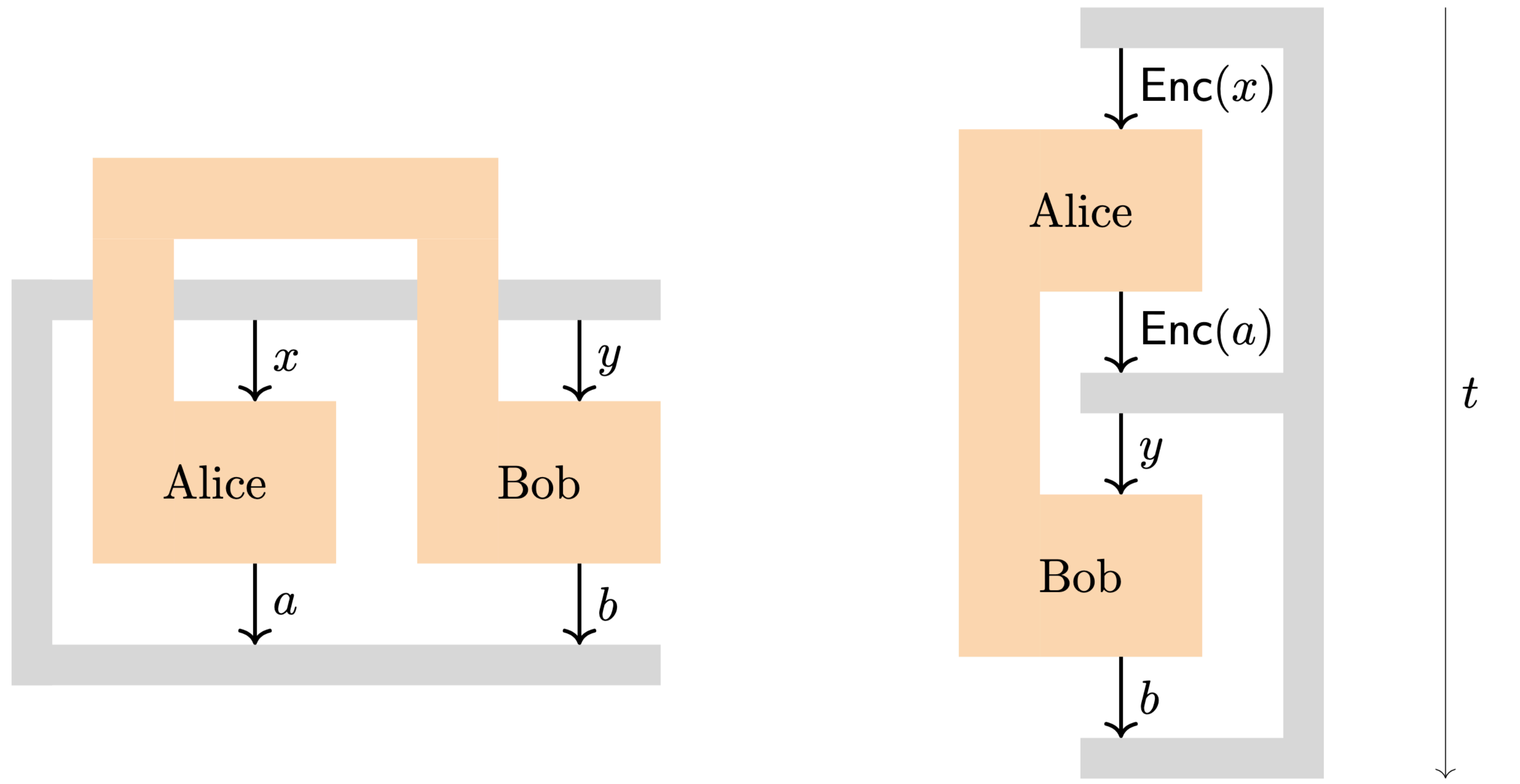

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Efficient provers cannot decrypt with more than negl(λ) probability

Previous results

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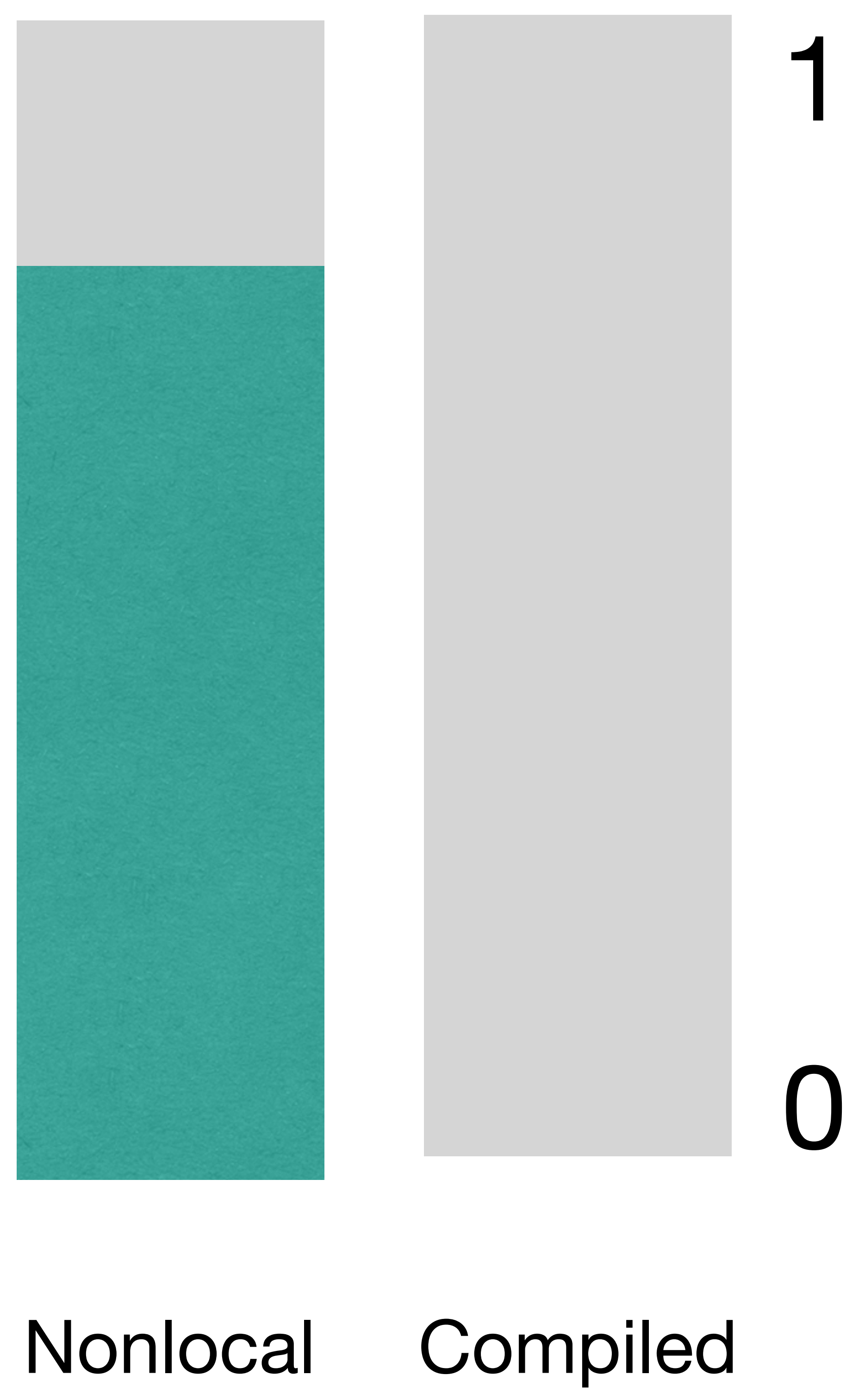
1. Classical soundness for all games [KLVY22]



[KLVY22] arXiv: 2203.15877

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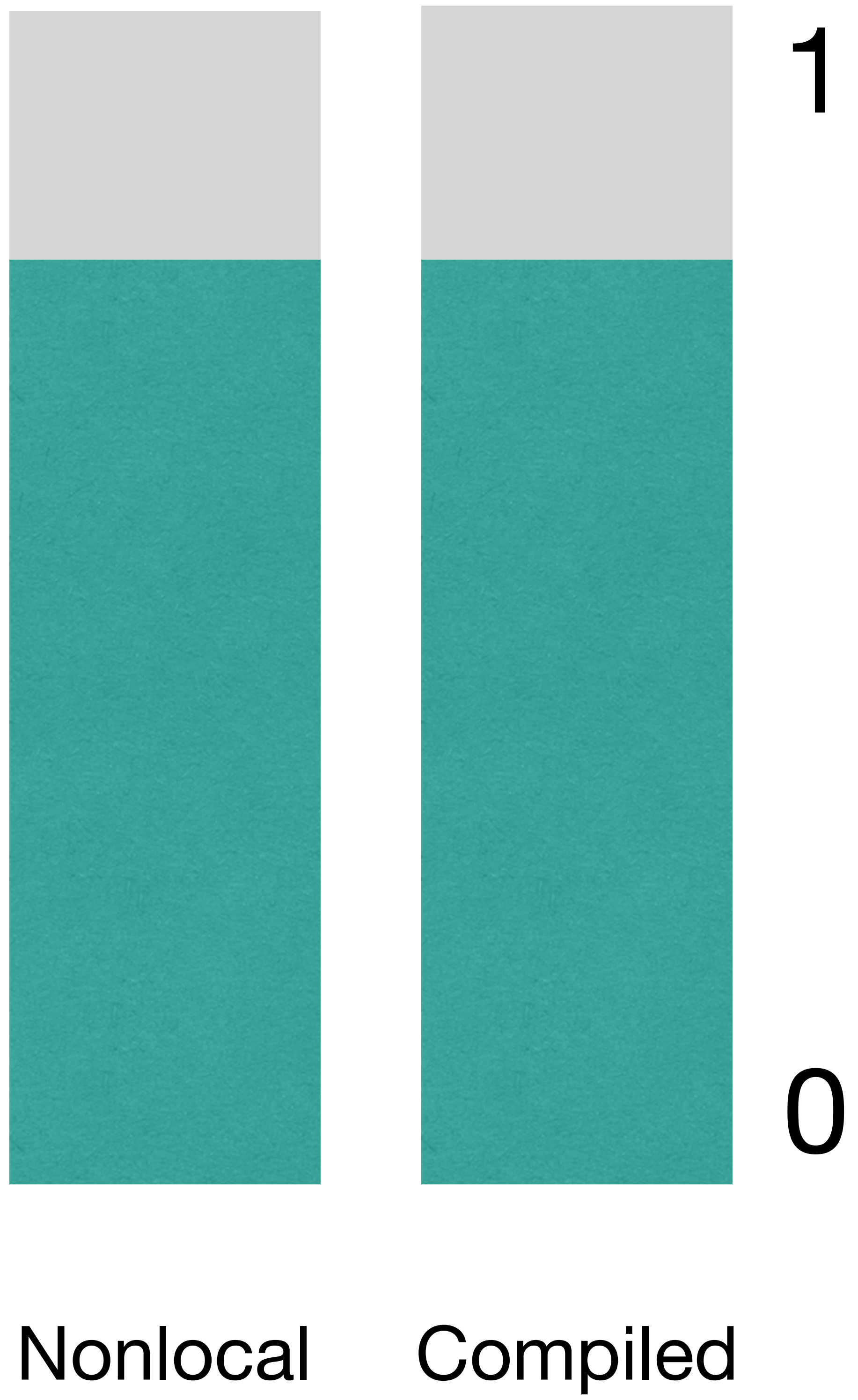

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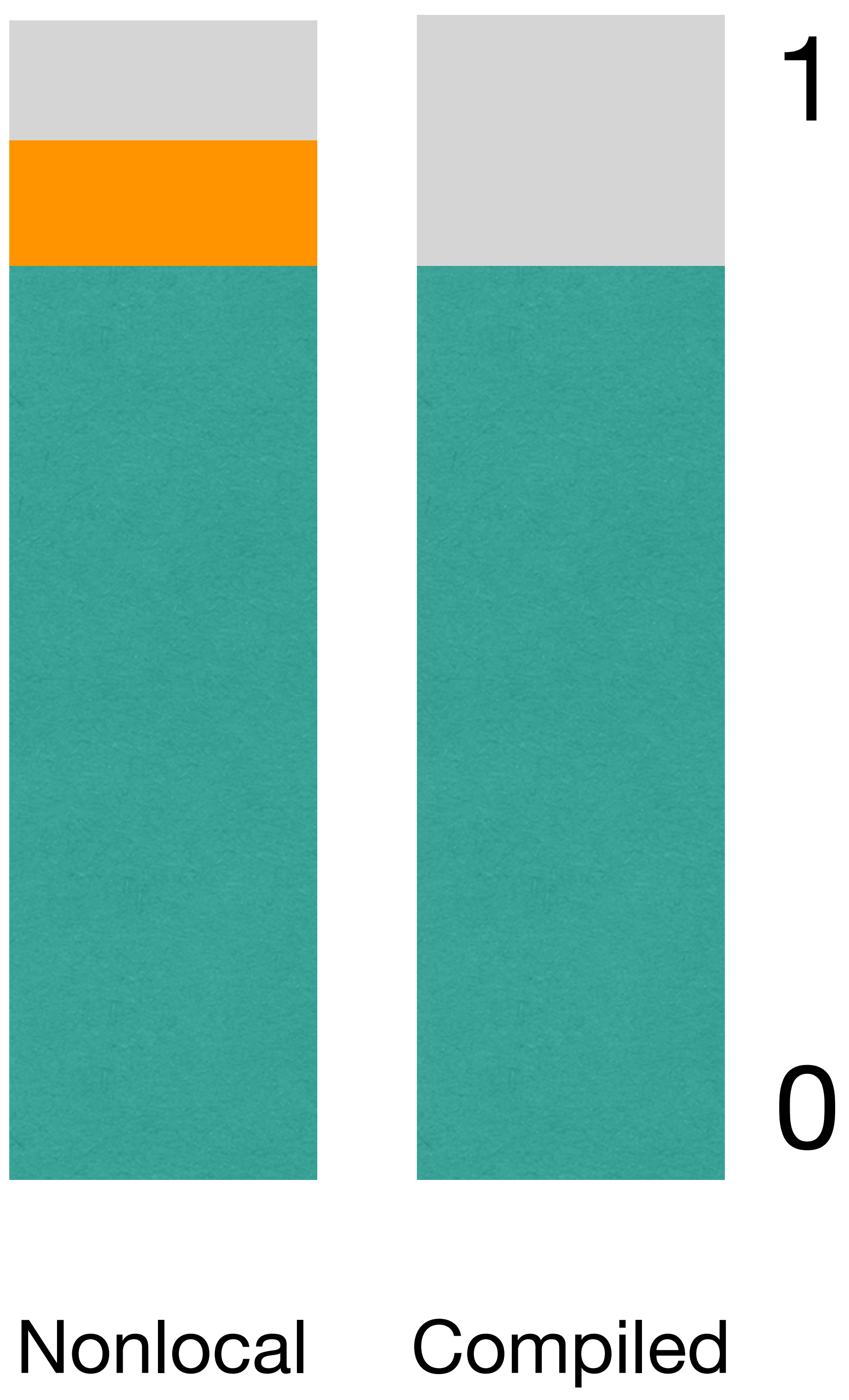

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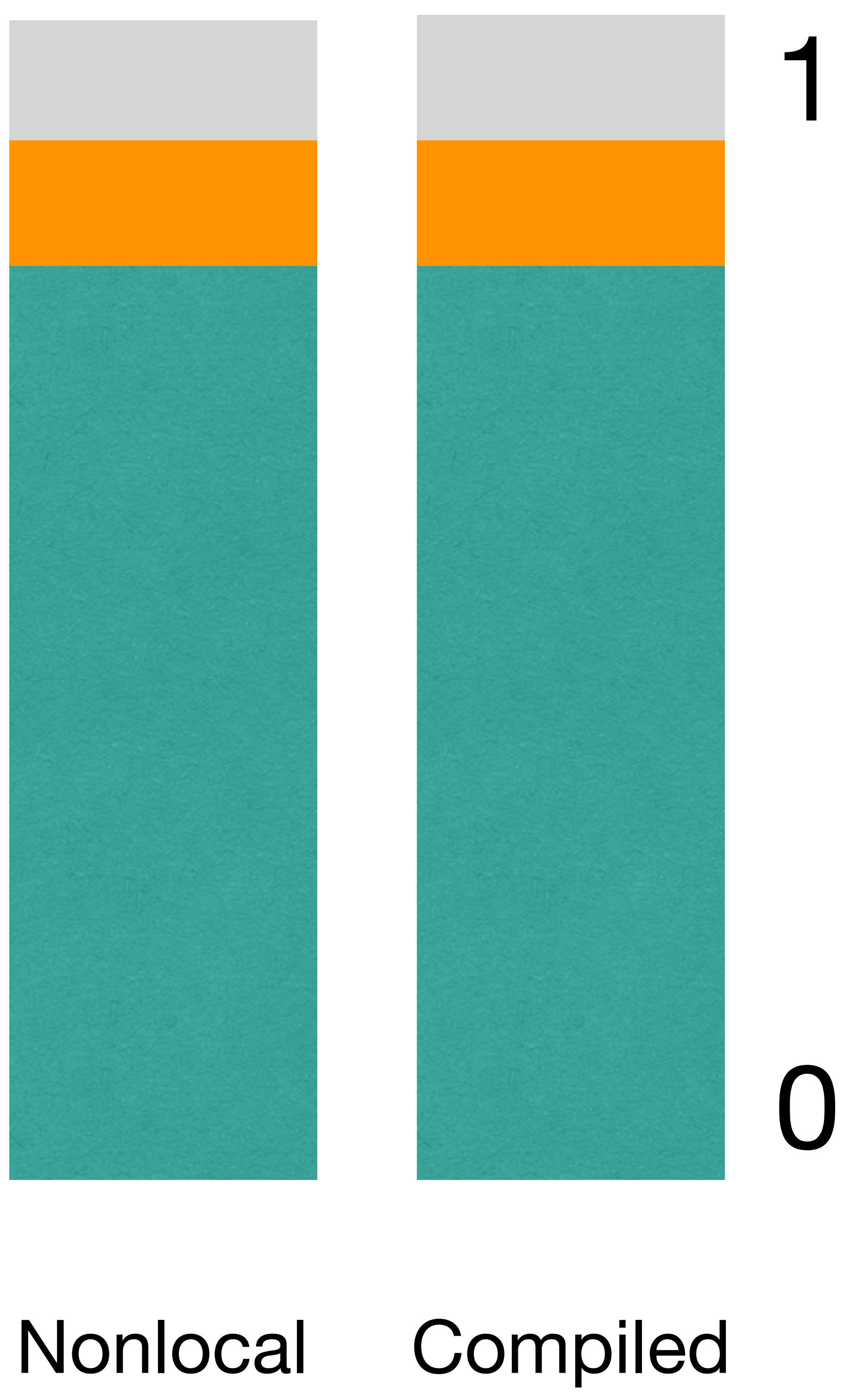

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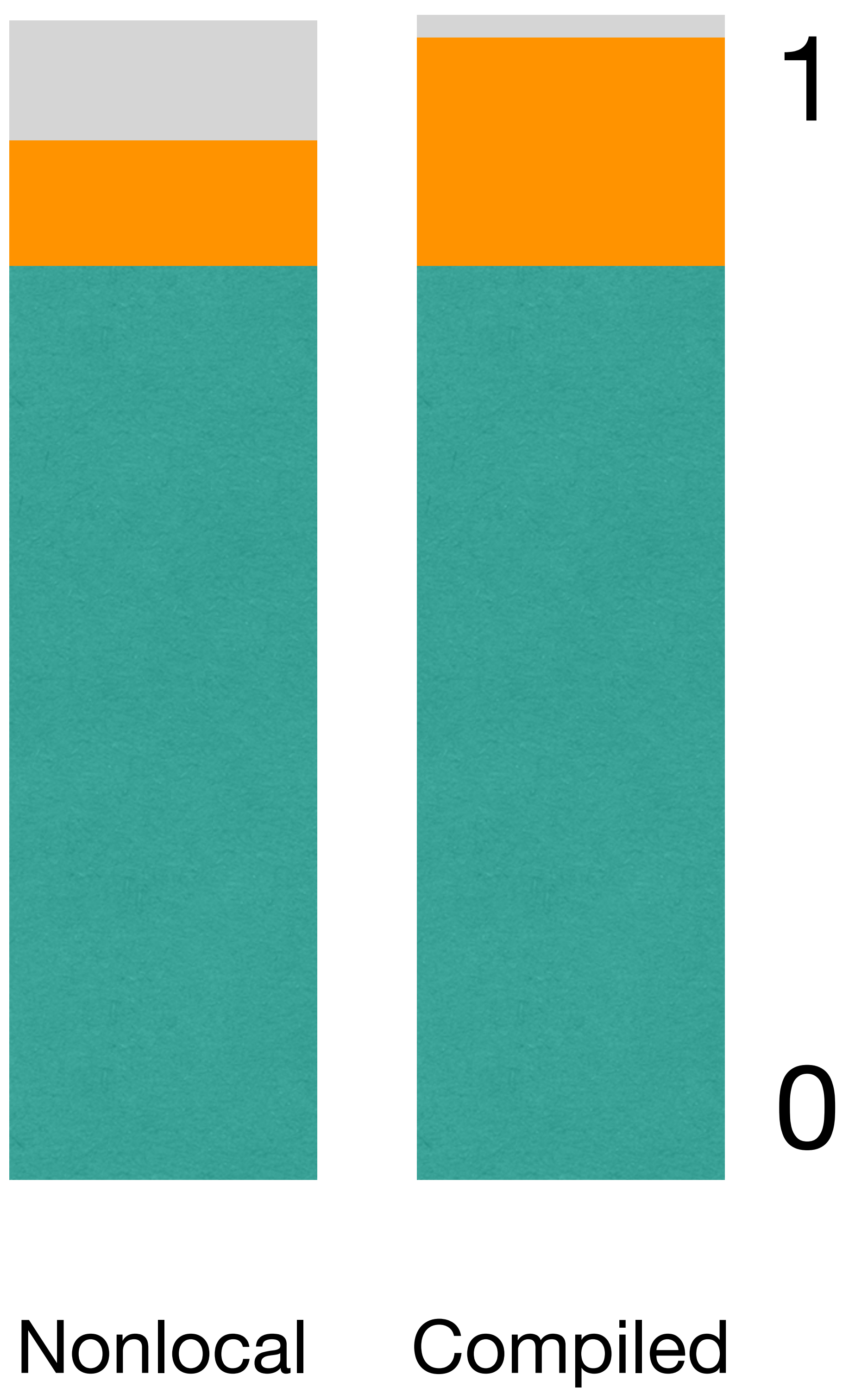

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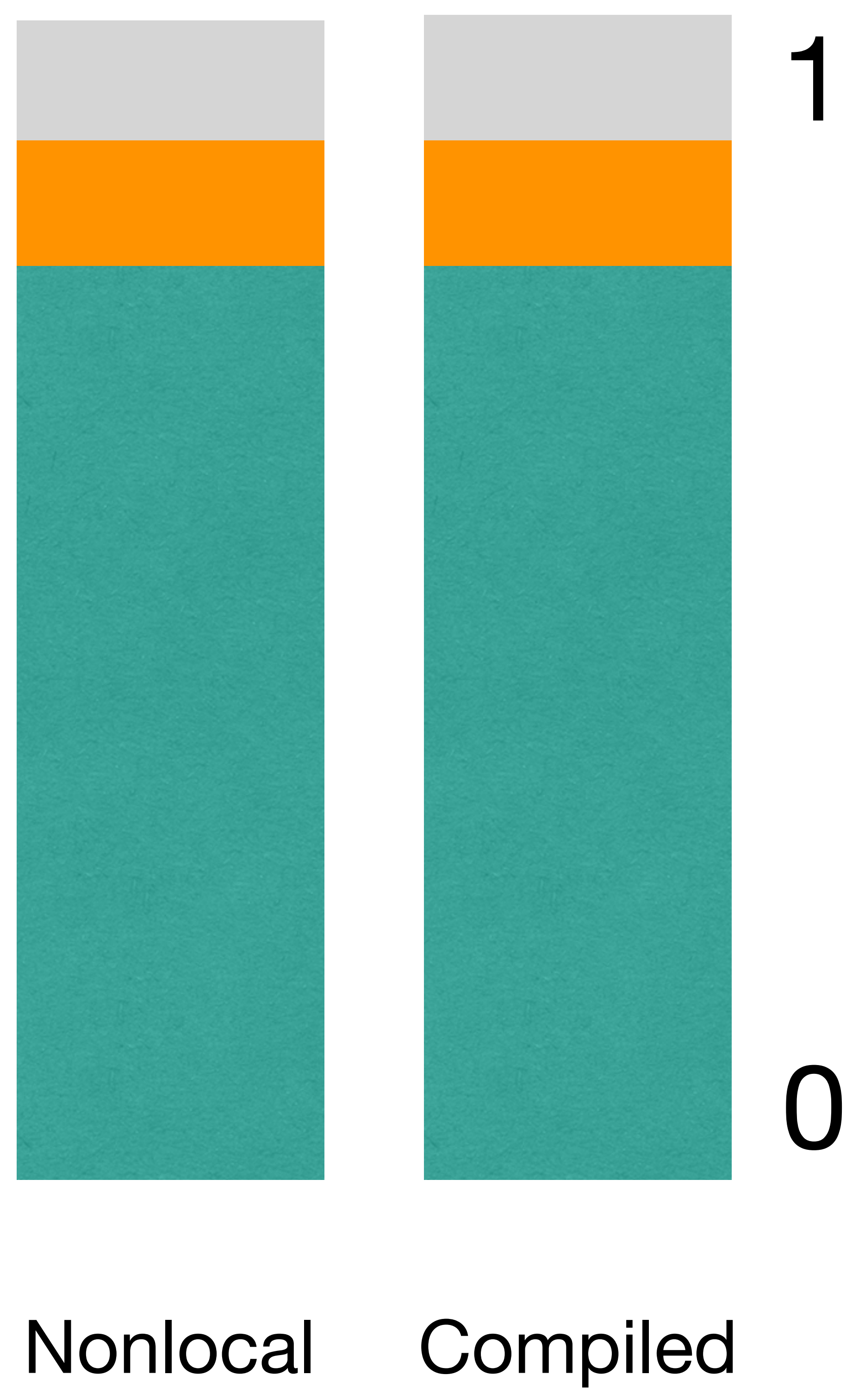
3. Quantum soundness for some bipartite games



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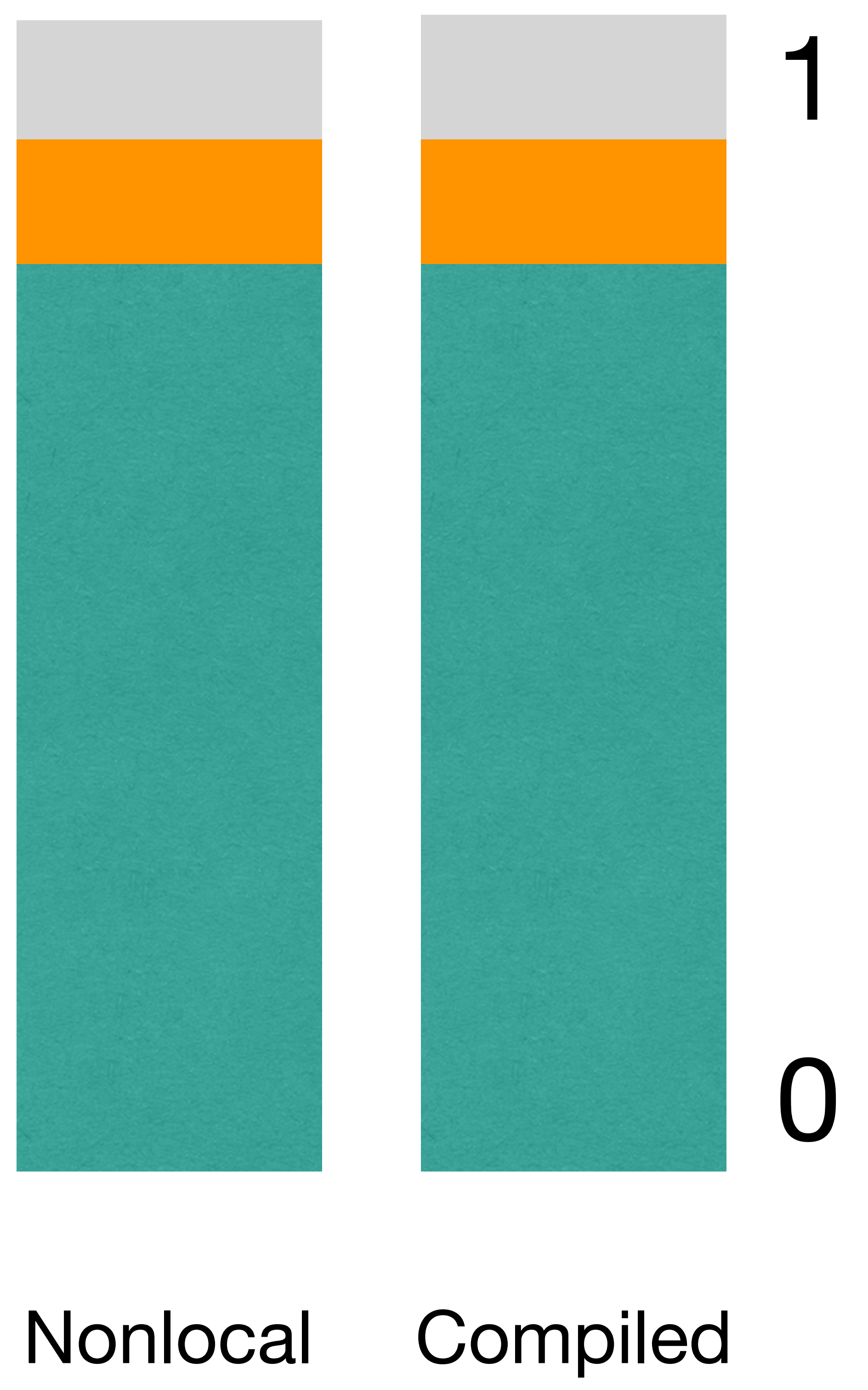
4. Asymptotic quantum soundness for all bipartite games [KMPSW24]



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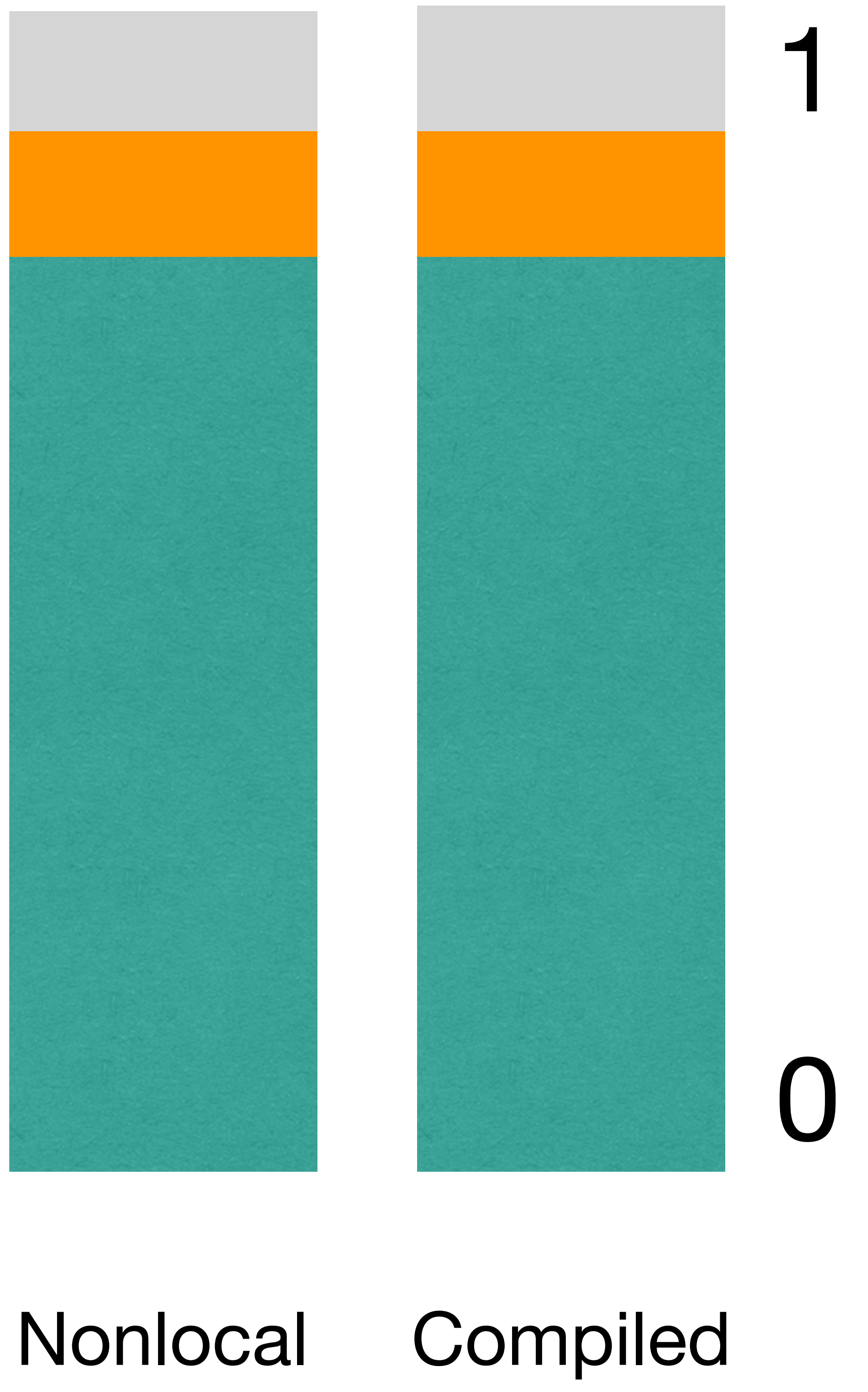
for the lack of better photo



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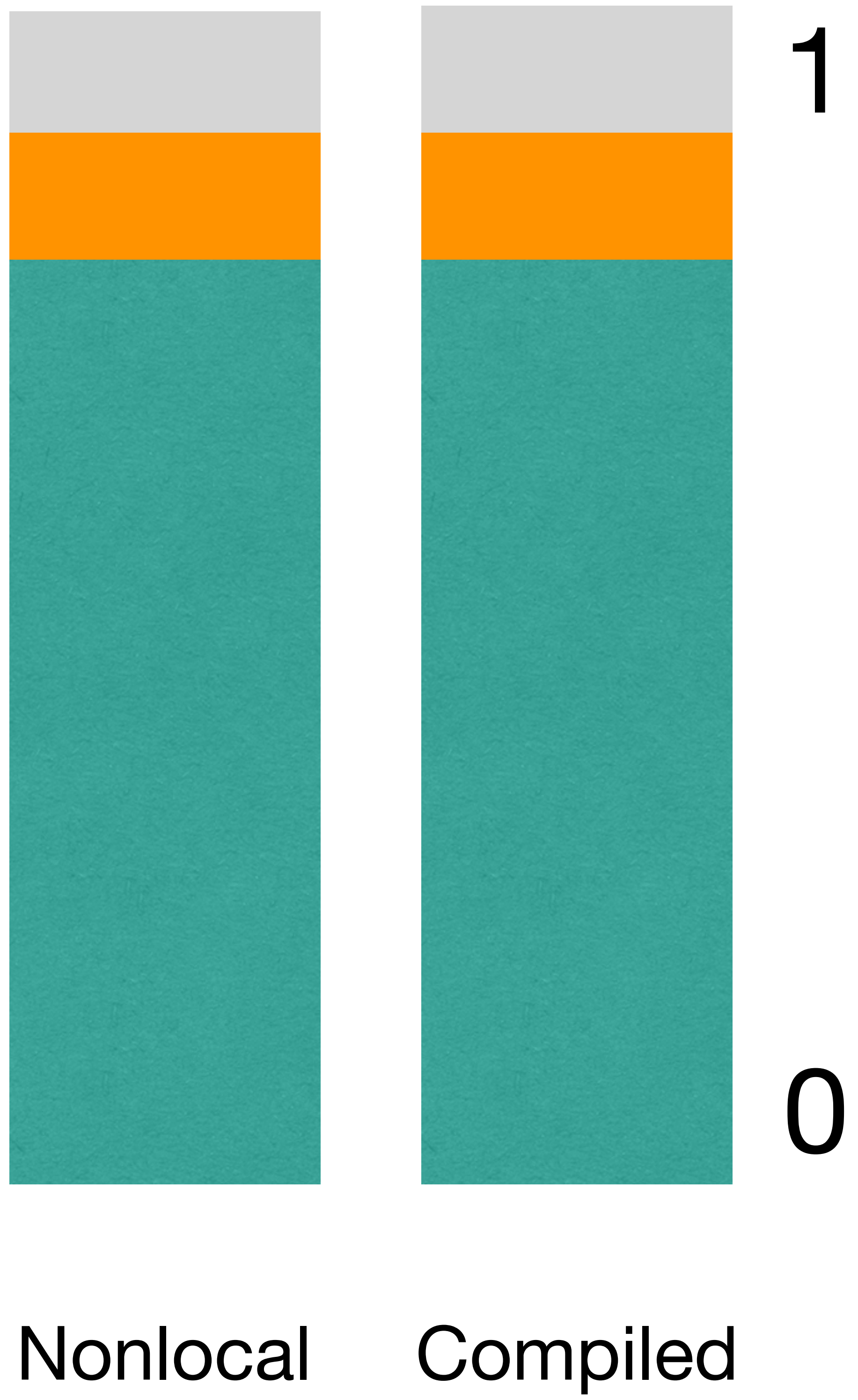

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Quantitative quantum soundness for bipartite compiled Bell games

Xiangling Xu (许湘灵), Inria Saclay

With Igor Kelp, Connor Paddock, Marc-Olivier Renou, Simon Schmidt, Lucas Tendick, Yuming Zhao

From KMPSW24 asymptotic to quantitative quantum soundness

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- Bipartite Bell game $\mathscr{G}$, compiled $\mathscr{G}_{\mathrm{comp}}$,
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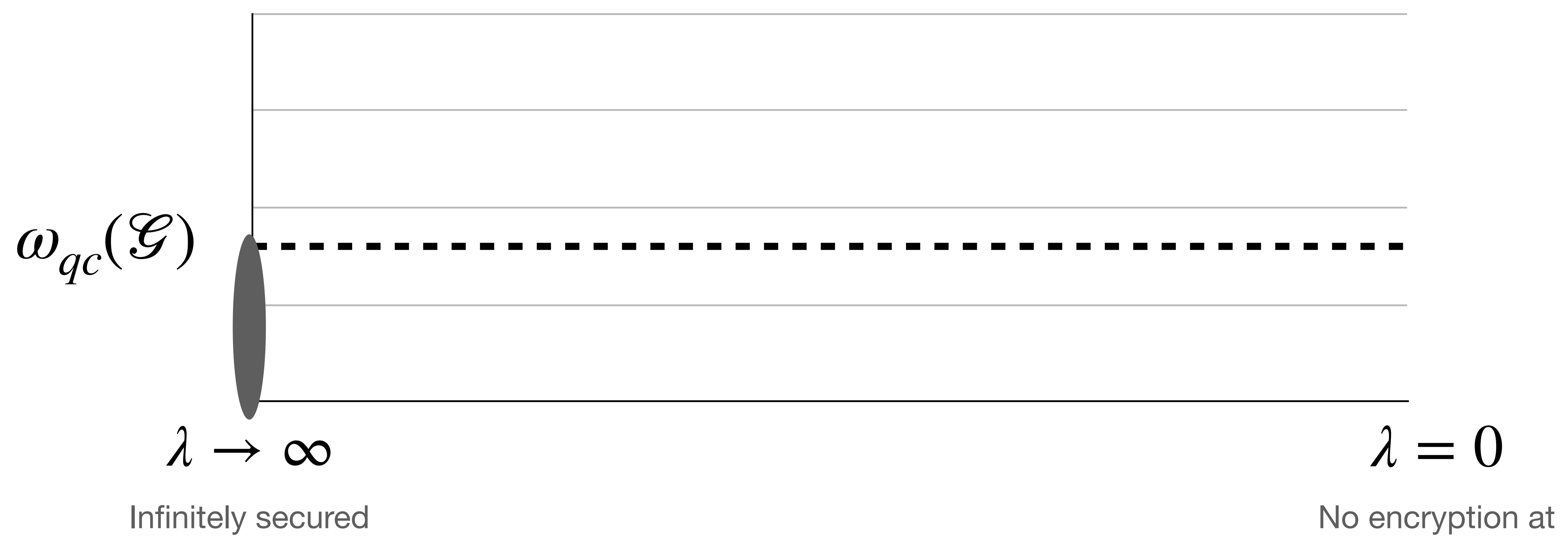

From KMPSW24 asymptotic to quantitative quantum soundness

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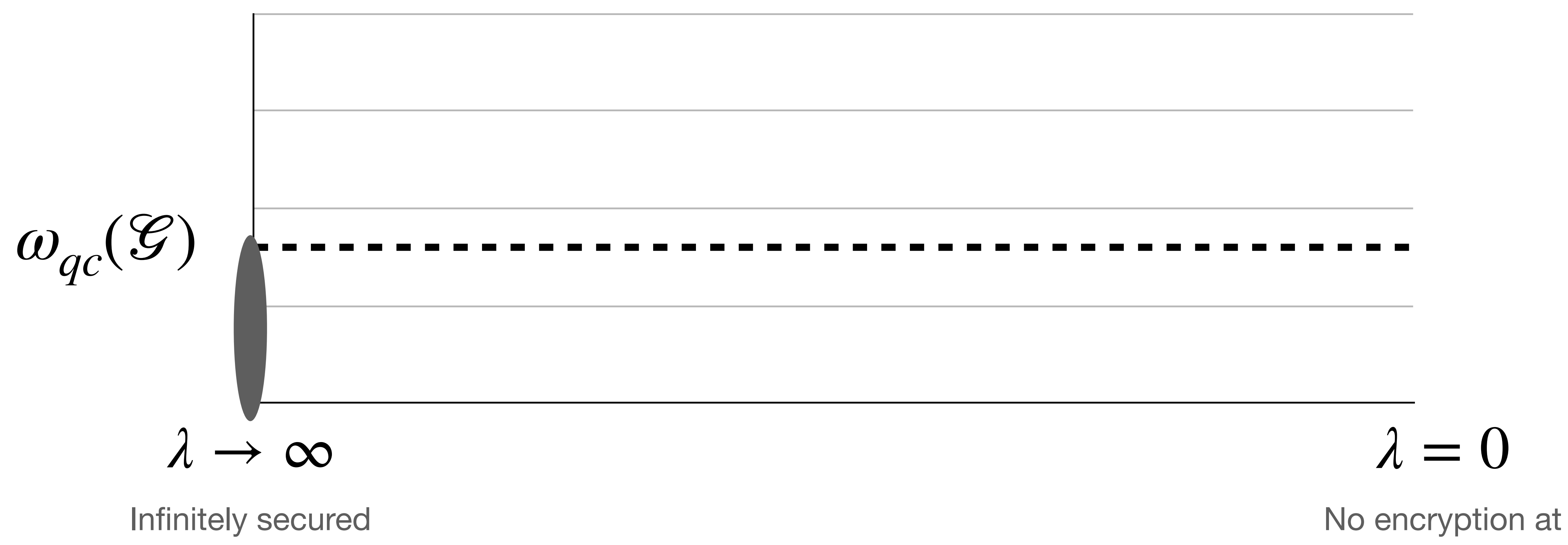
- Asymptotically sound, the score:
$$\lim_{\lambda \to \infty} \omega_\lambda(\mathscr{G}_{\text{comp}}, S) \le \omega_{qc}(\mathscr{G})$$



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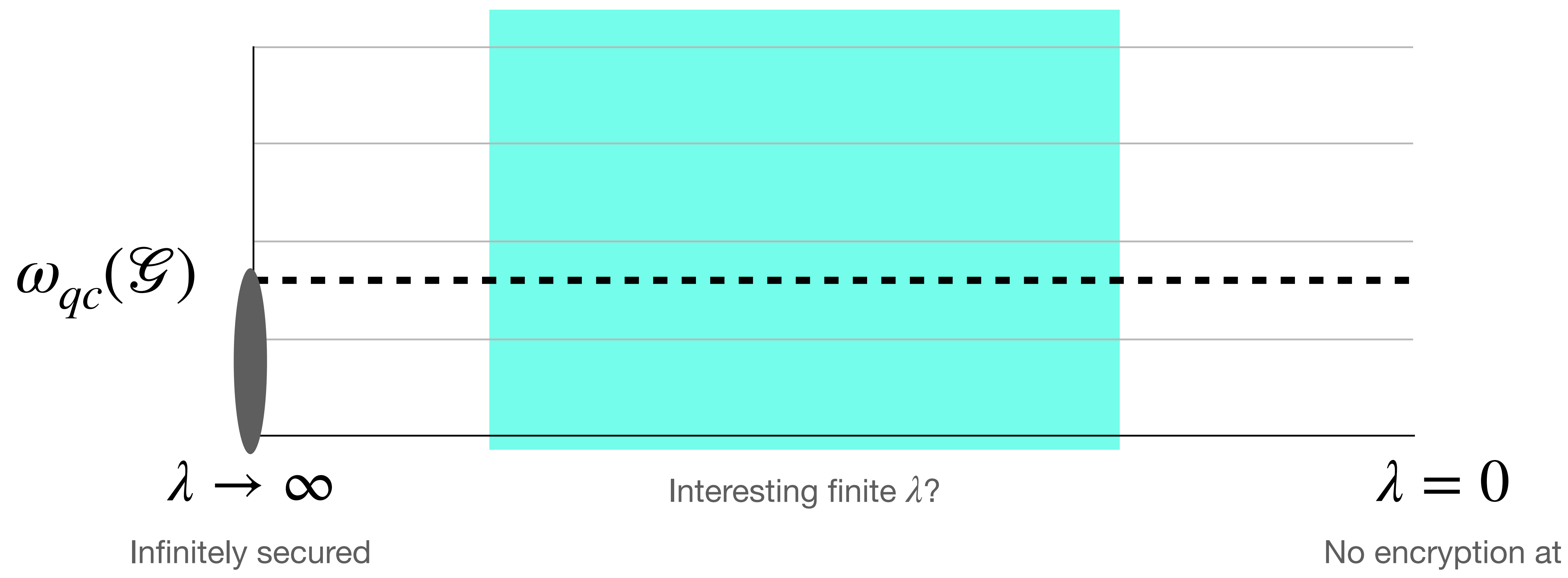

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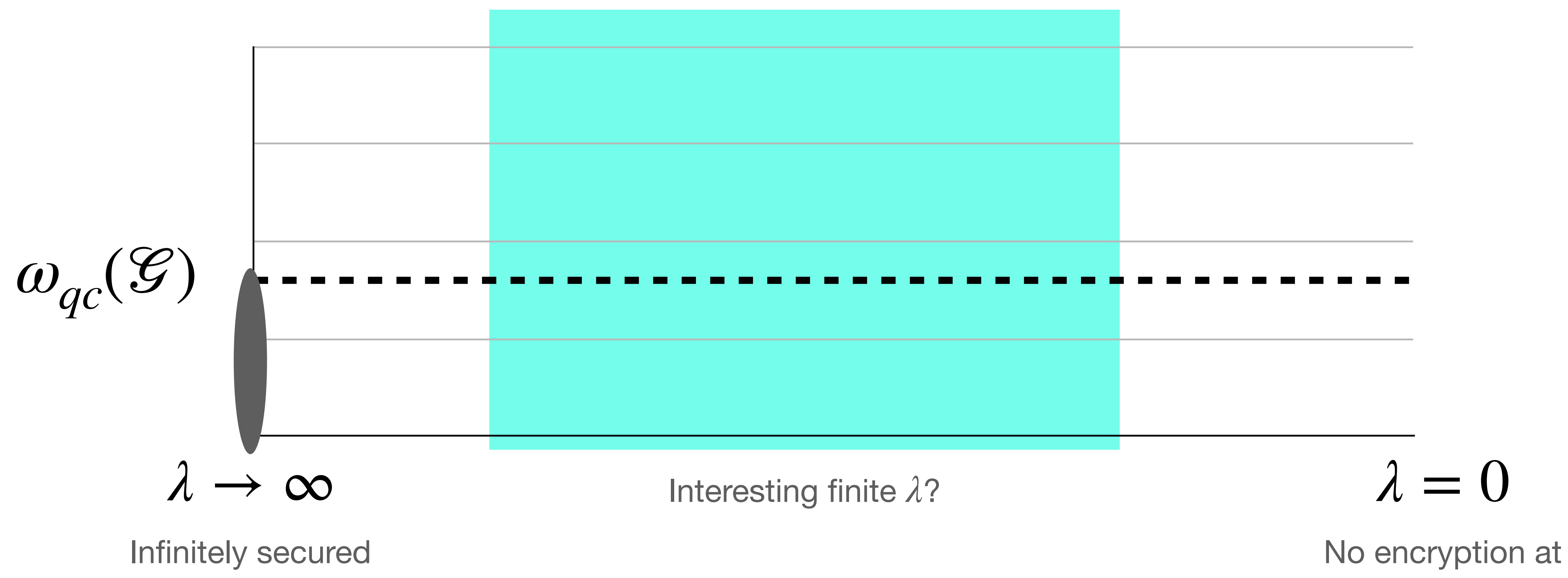

- But how much more a cheating prover can win with S_λ at *finite* λ?

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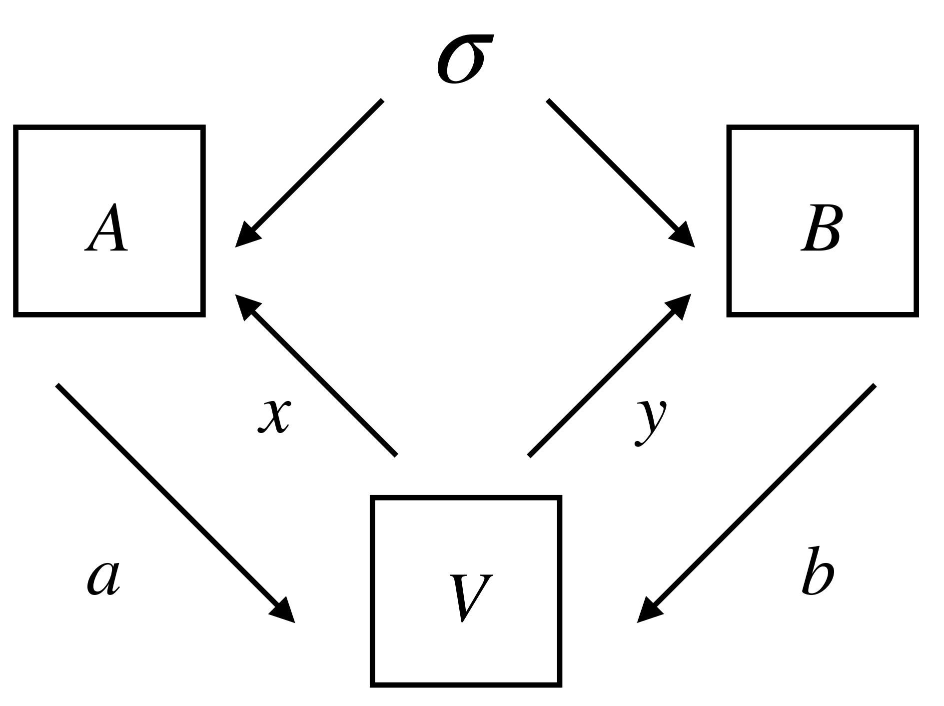
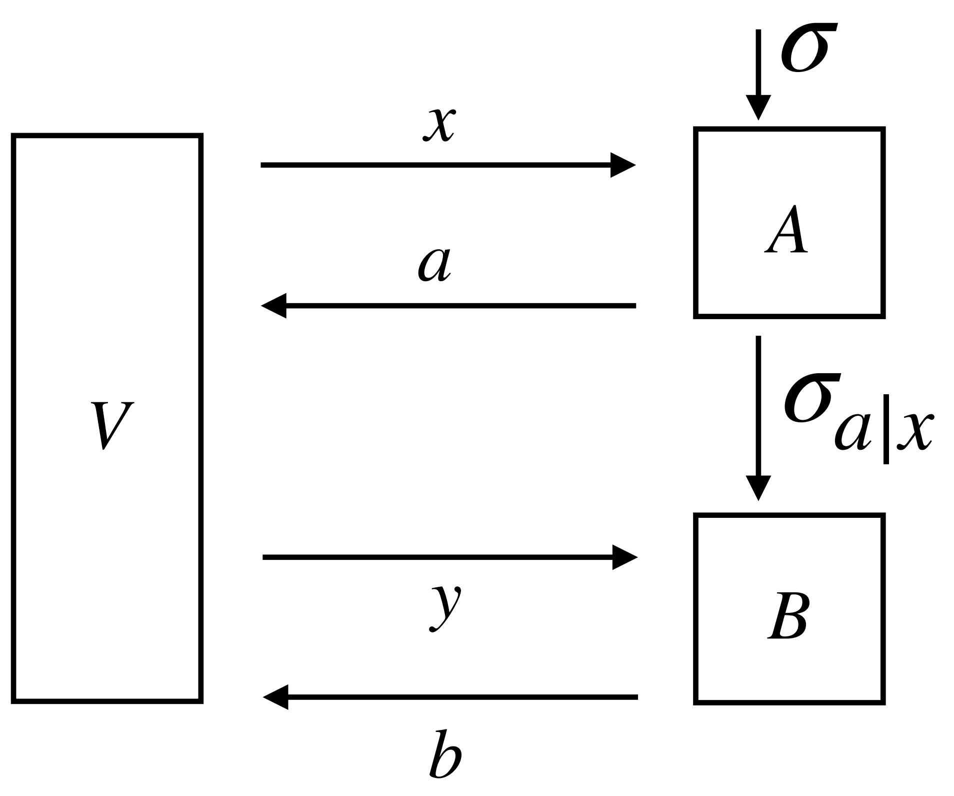
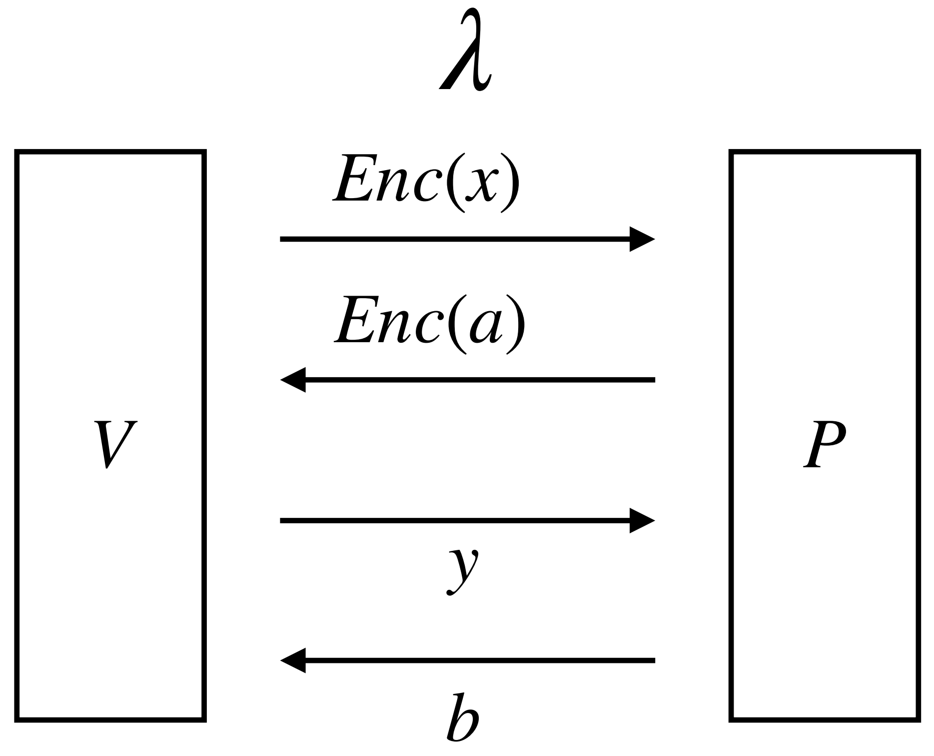
- $\epsilon_{\text{seqNPA}}(n)$ ~ approximation error of a novel *sequential NPA hierarchy*
- Plan: Recap of asymptotic paper, then sequential NPA, then main result revisit

Nonlocal to sequential to compiled







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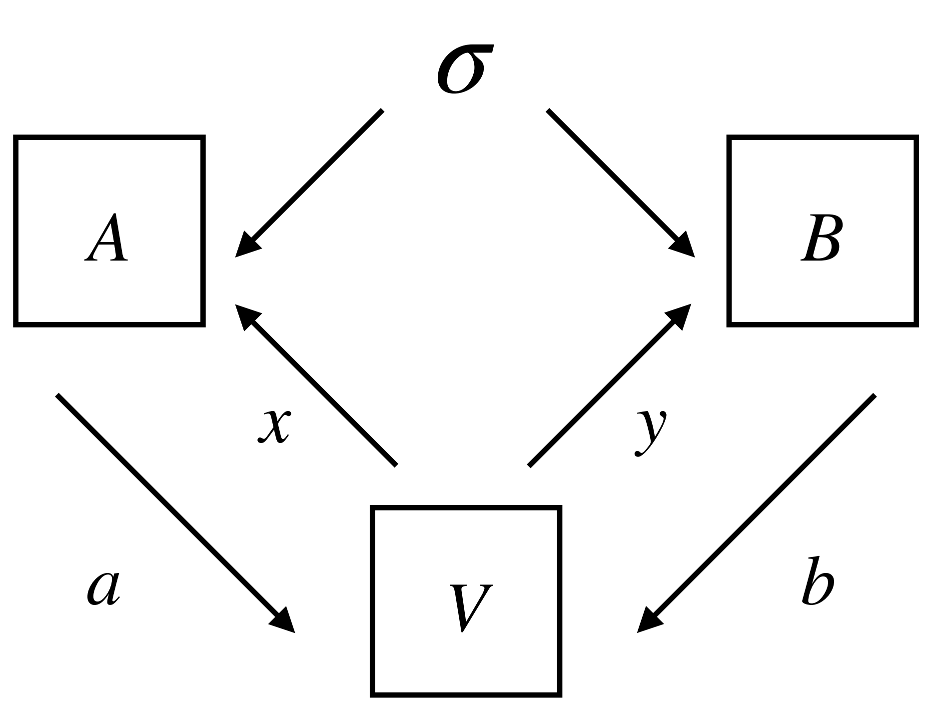
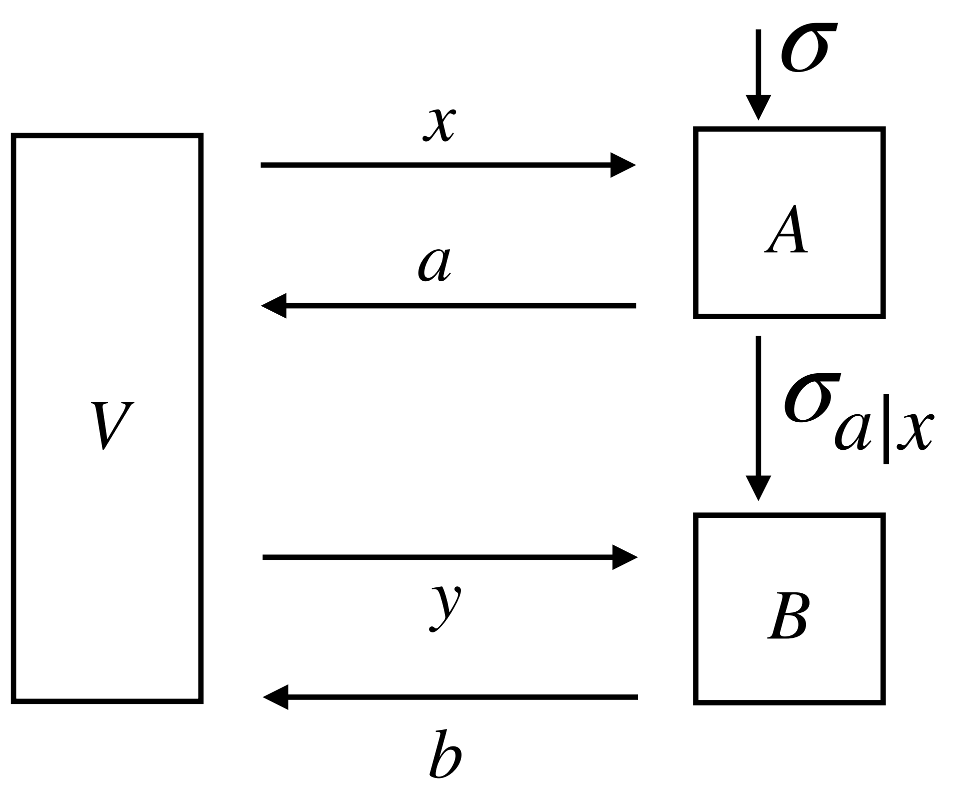
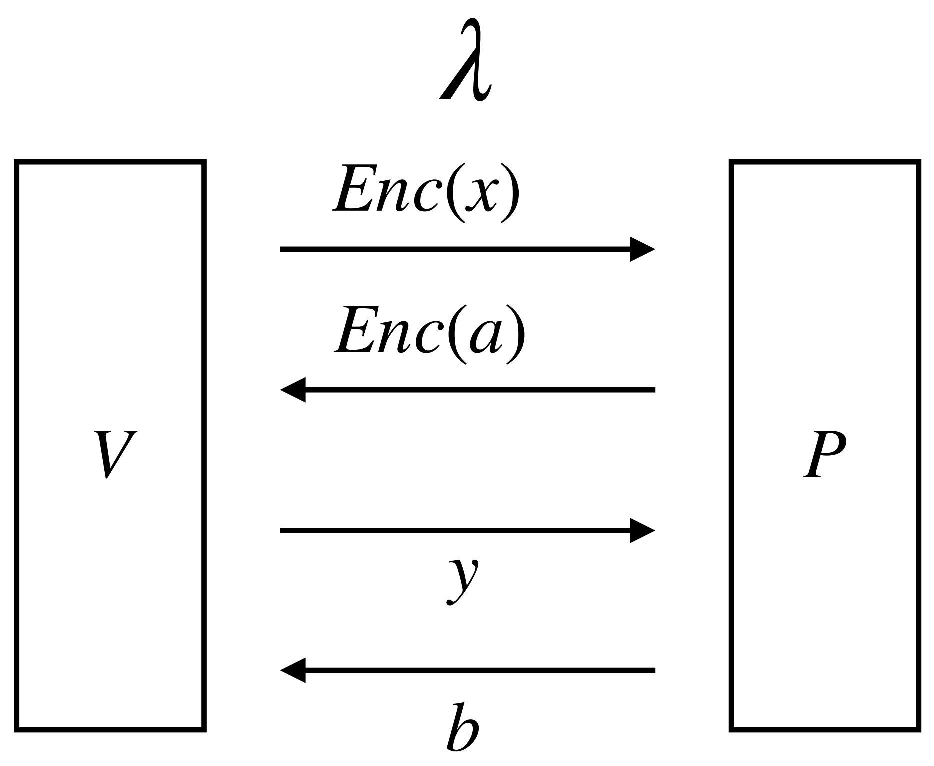
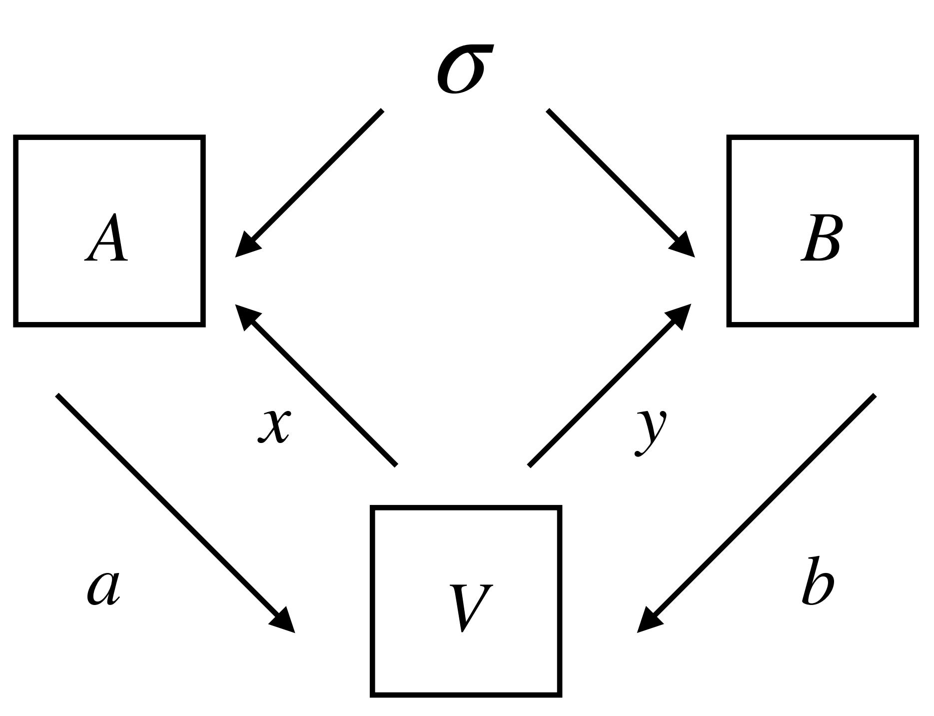


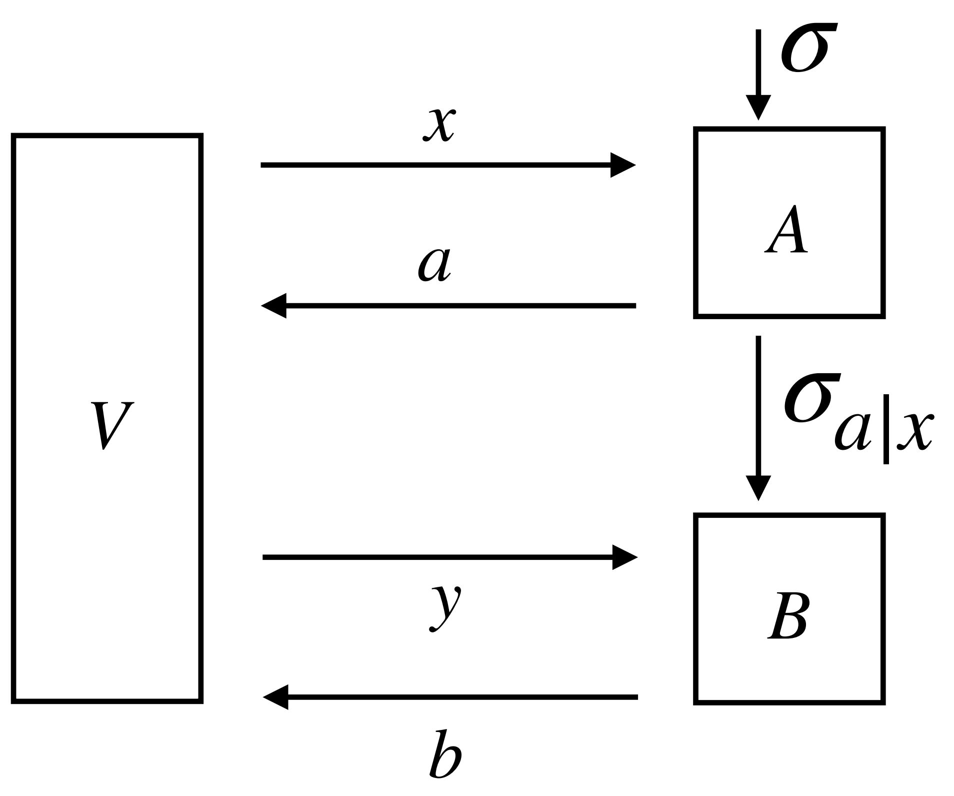


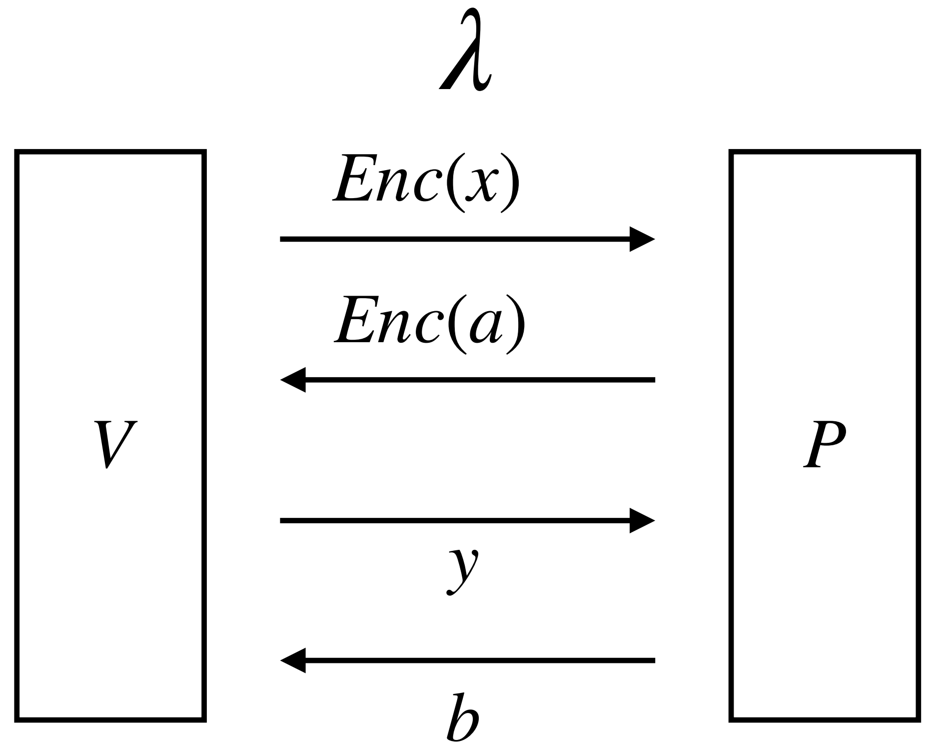


- σ density operator

Nonlocal to sequential to compiled







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- $A_{a|x}, B_{b|y}$ POVMs ($\sum_a A_{a|x} = 1$)

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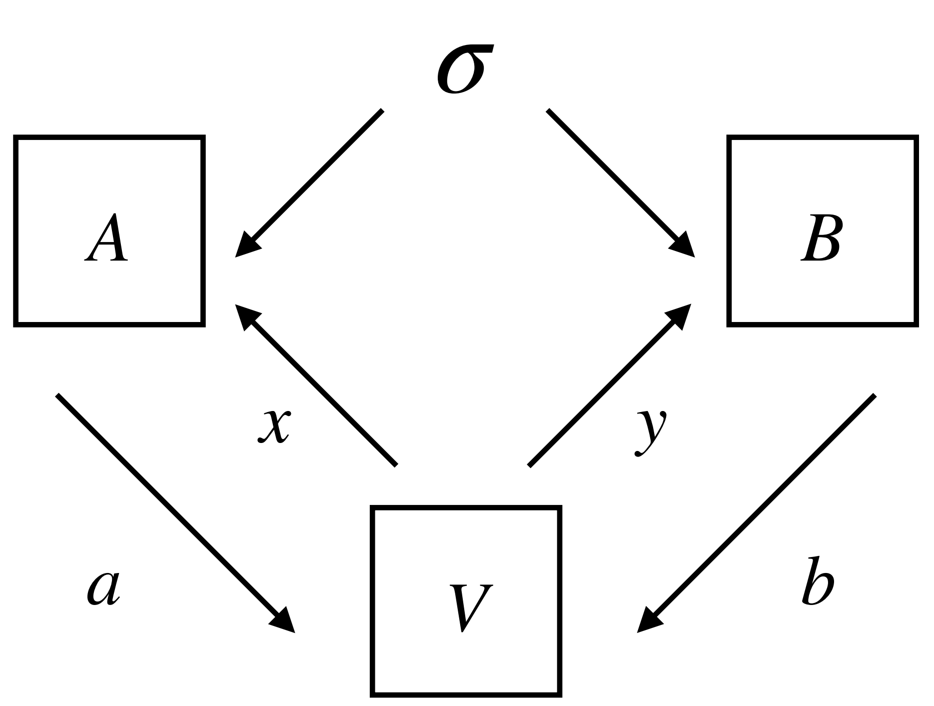

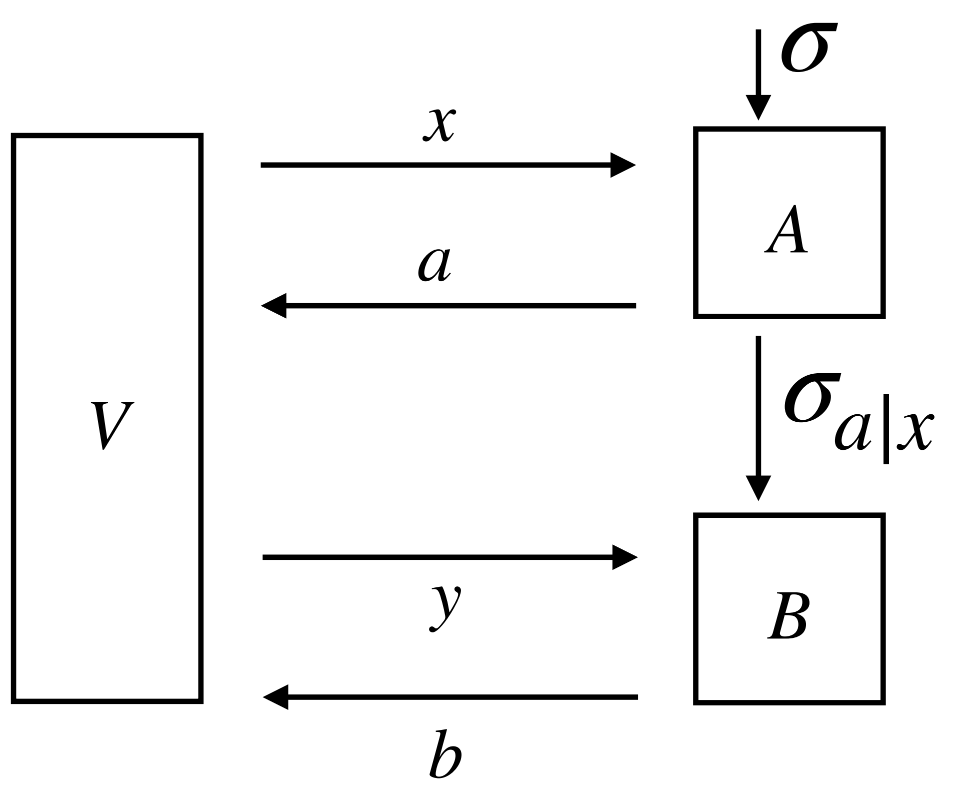

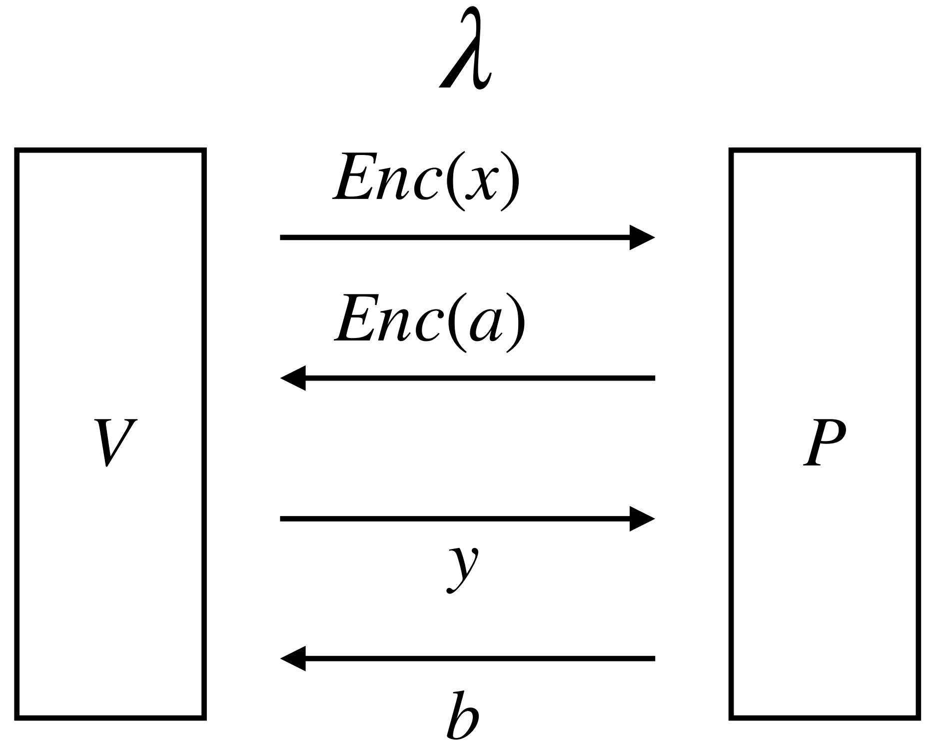

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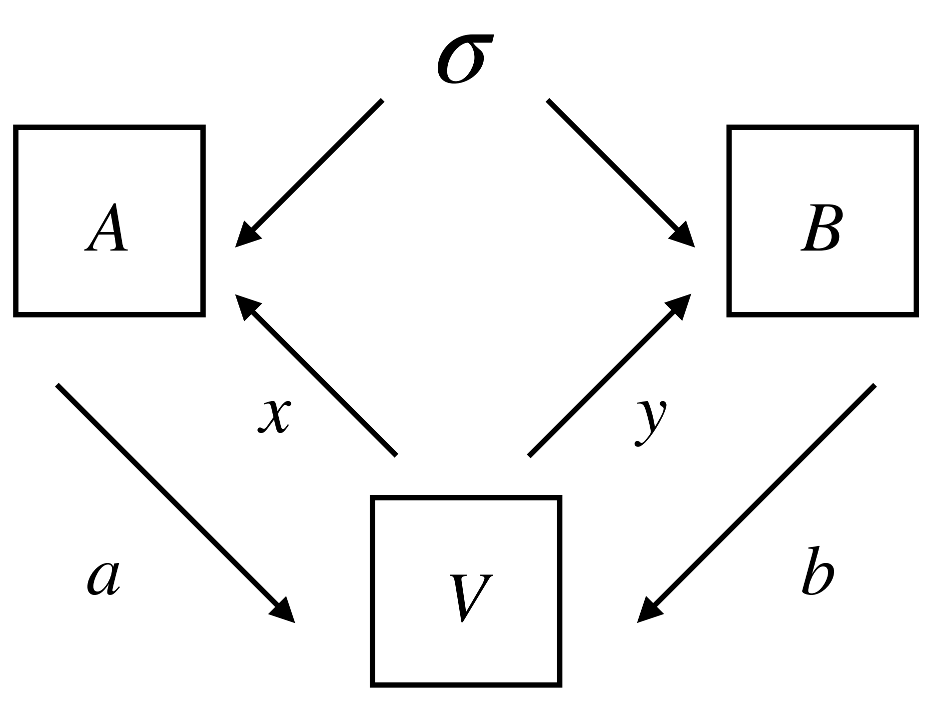
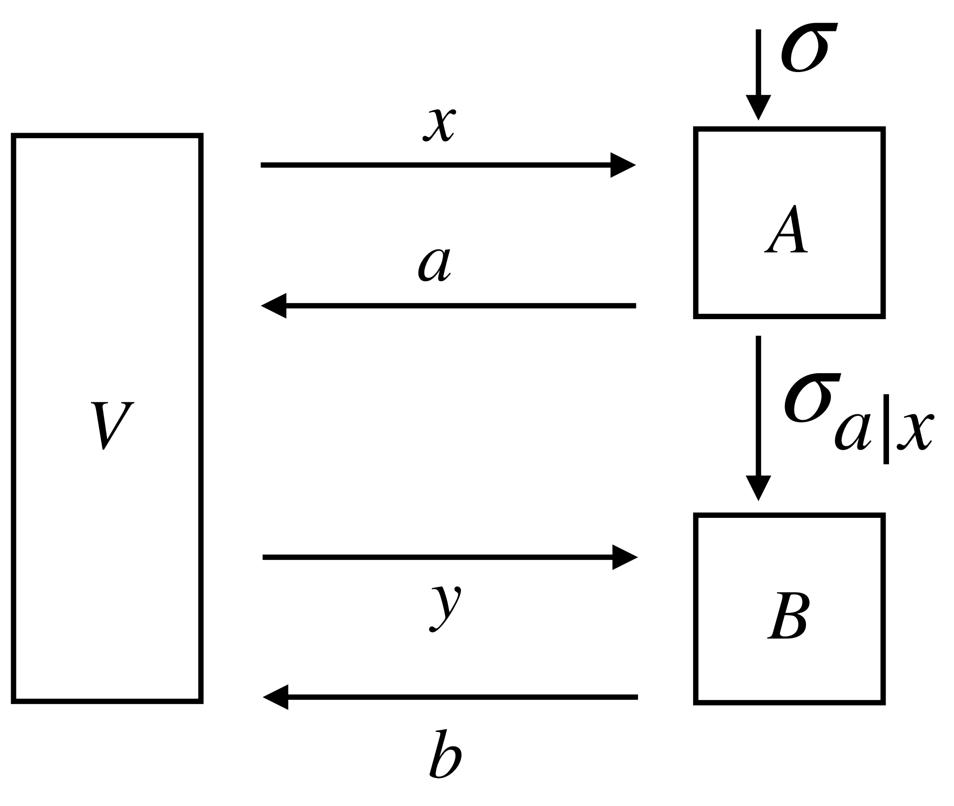
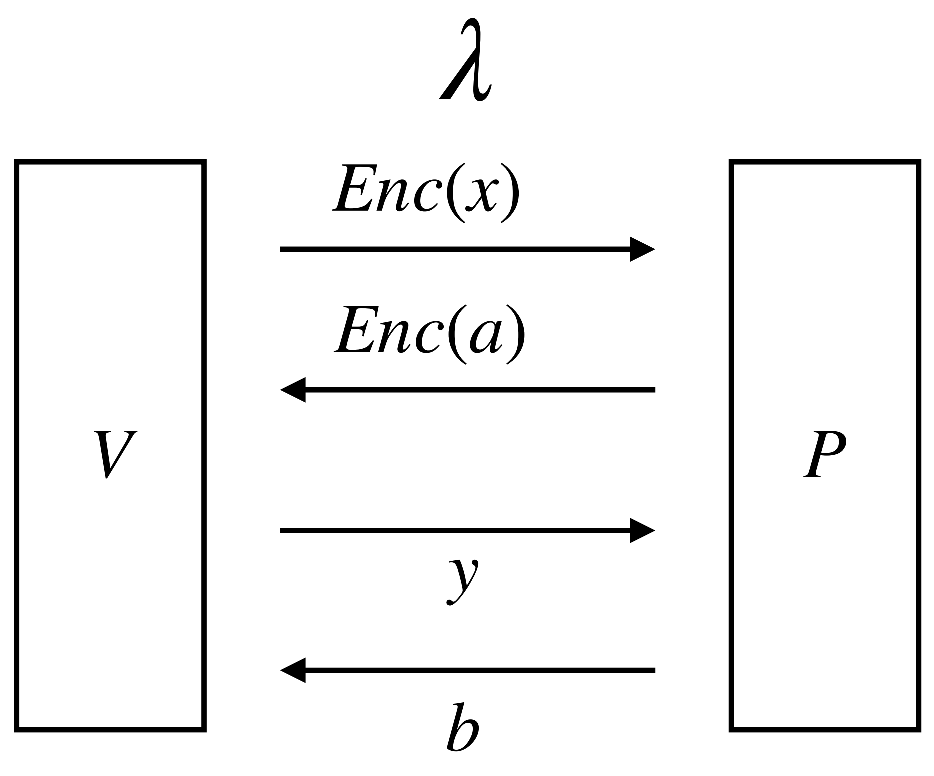
- $A_{a|x}, B_{b|y}$ POVMs ($\sum_a A_{a|x} = 1$)
- $p(ab\,|\,xy) = \mathrm{Tr}(\sigma A_{a|x} \otimes B_{b|y}) := \sigma(A_{a|x}B_{b|y})$

Nonlocal to sequential to compiled



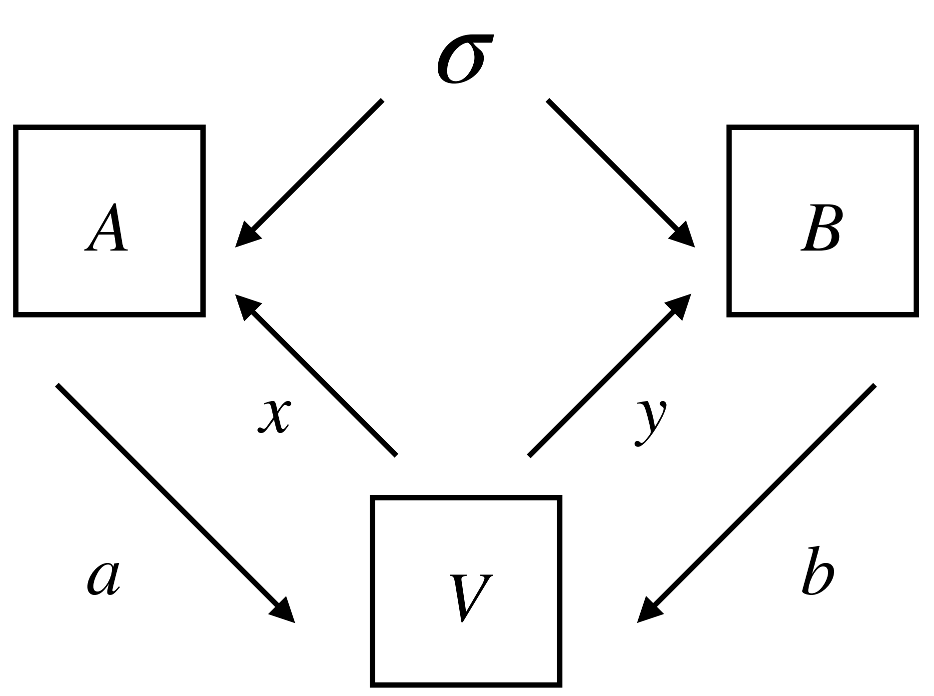


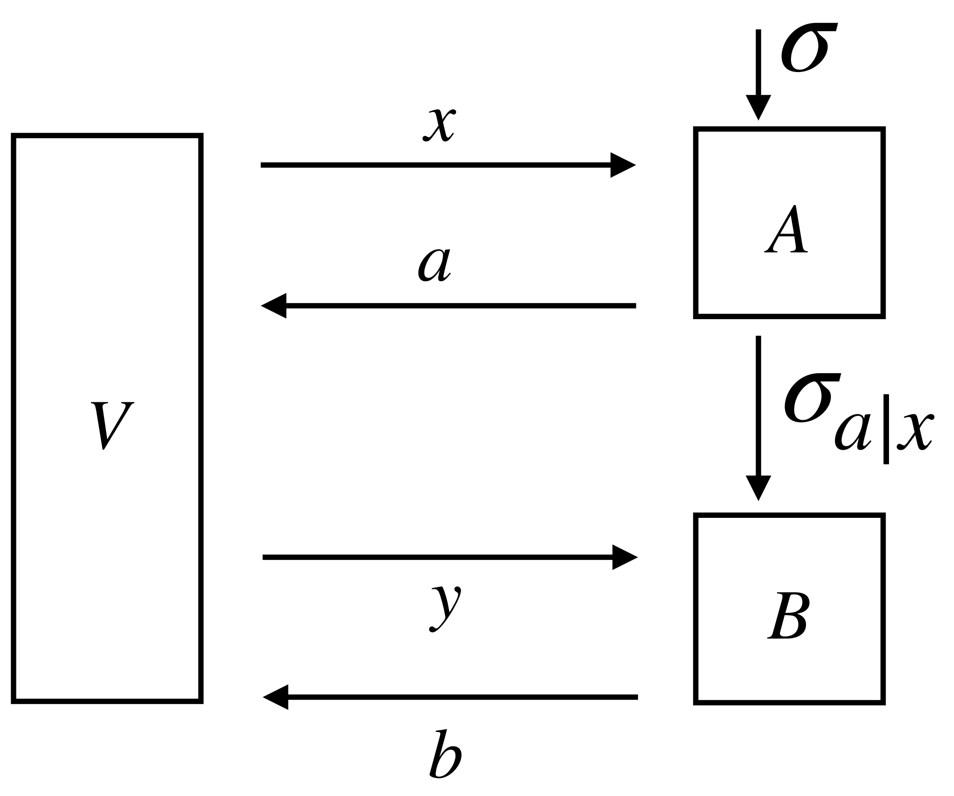


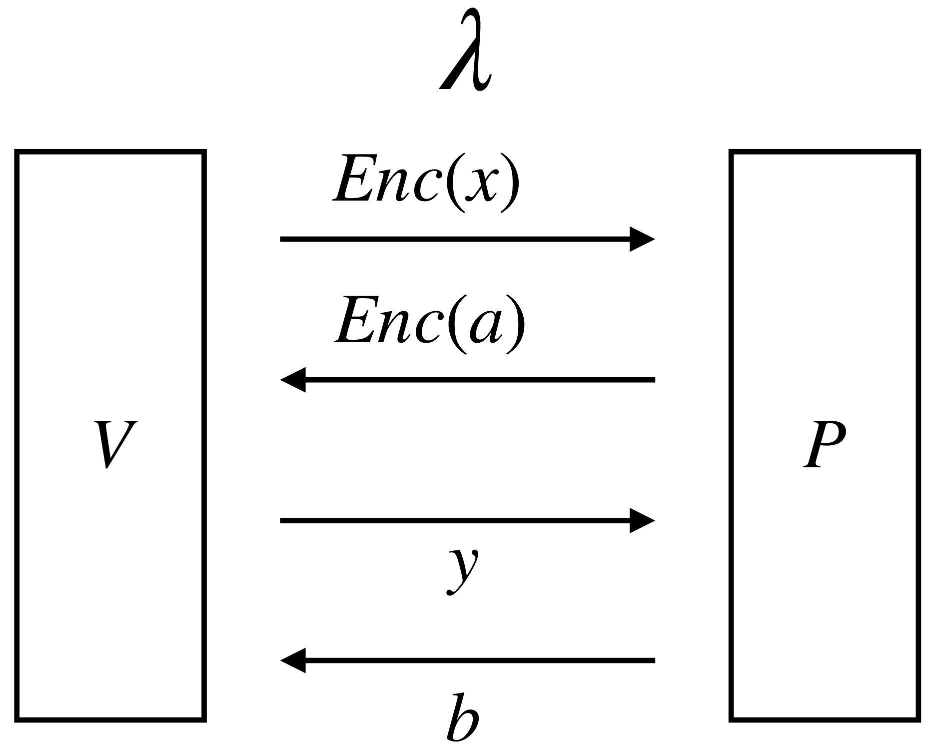
- σ density operator
- $A_{a|x}, B_{b|y}$ POVMs ($\sum_a A_{a|x} = 1$)
- $p(ab\,|\,xy) = \mathrm{Tr}(\sigma A_{a|x} \otimes B_{b|y}) := \sigma(A_{a|x}B_{b|y})$

Treat σ as positive linear functional

Nonlocal to sequential to compiled

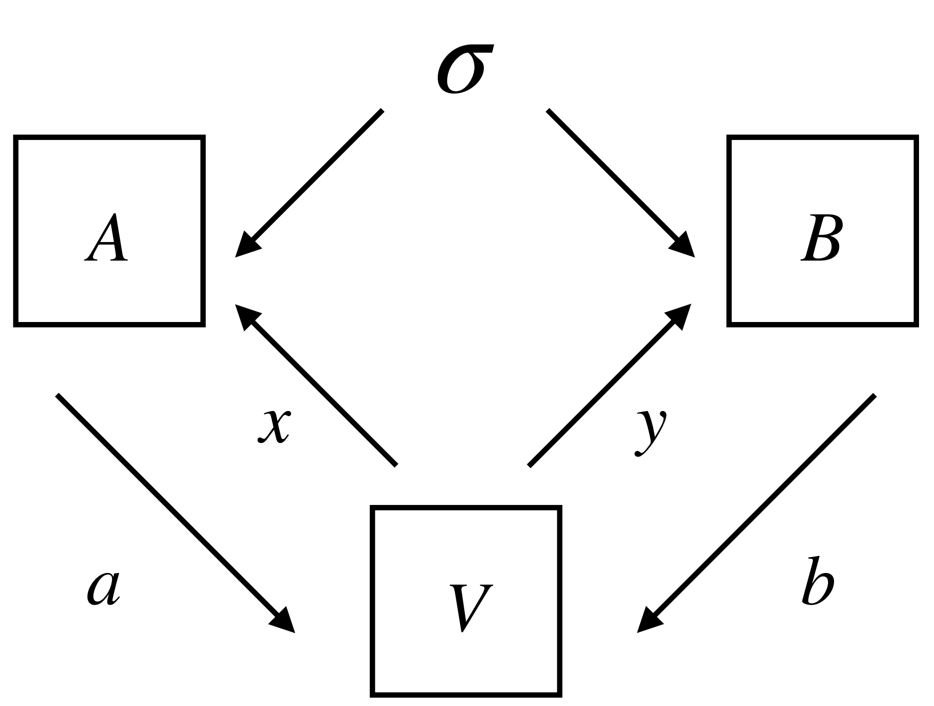


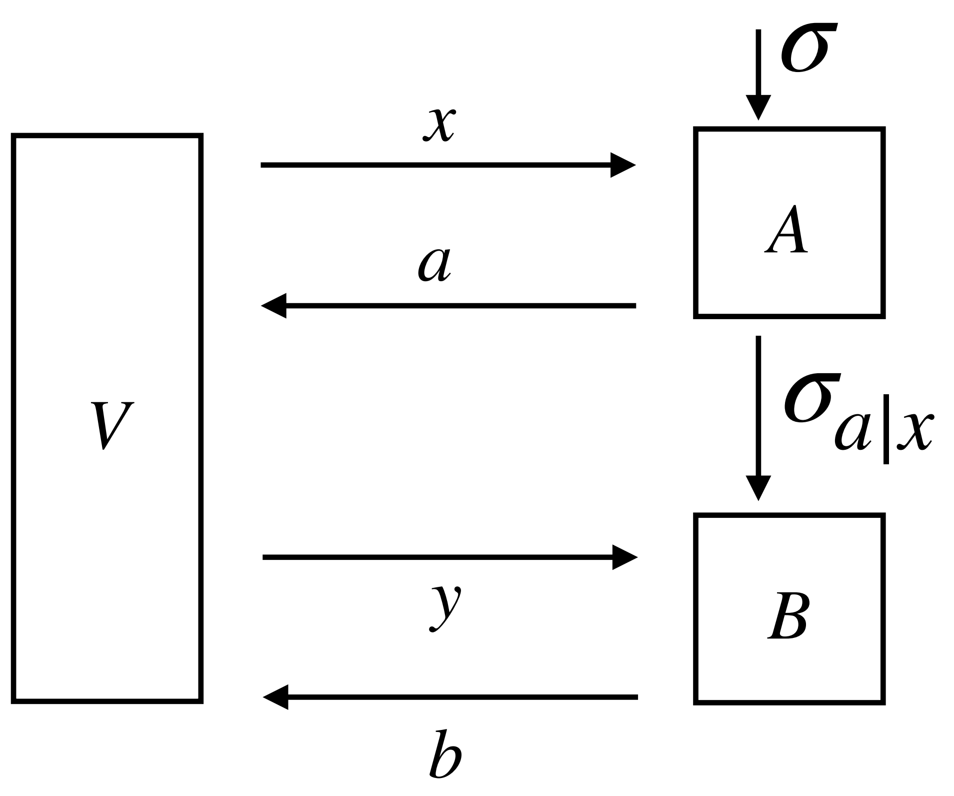


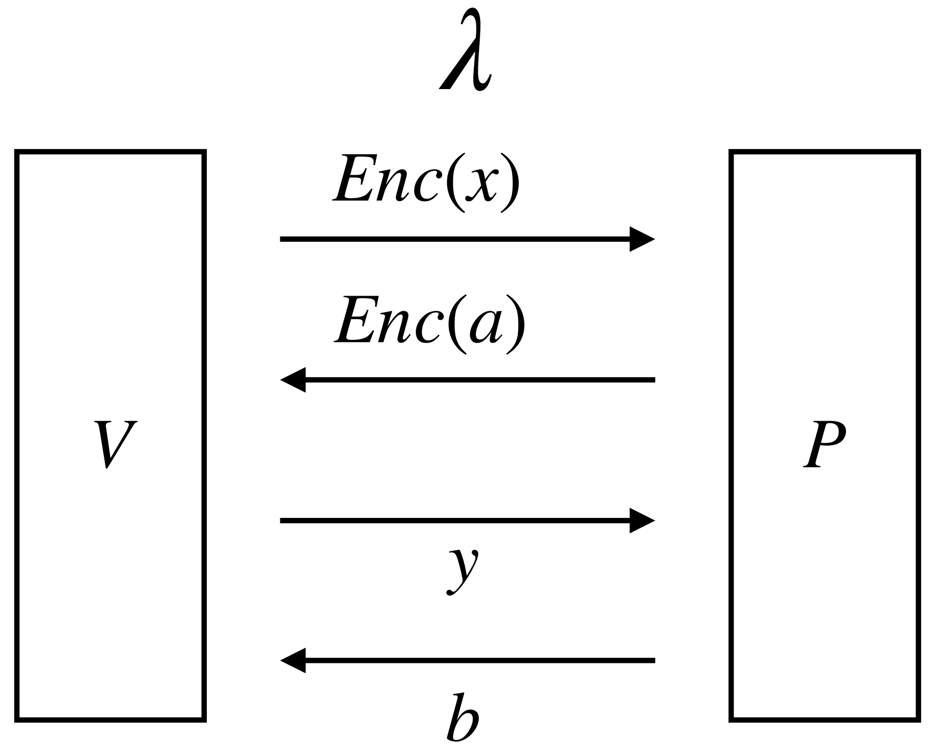


- $\sum_a A_{a|x} = 1$
- $p(ab\,|\,xy) := \sigma(A_{a|x}B_{b|y})$

Nonlocal to sequential to compiled







- Alice first measures and send the post-measured state $\sigma_{a|x}$

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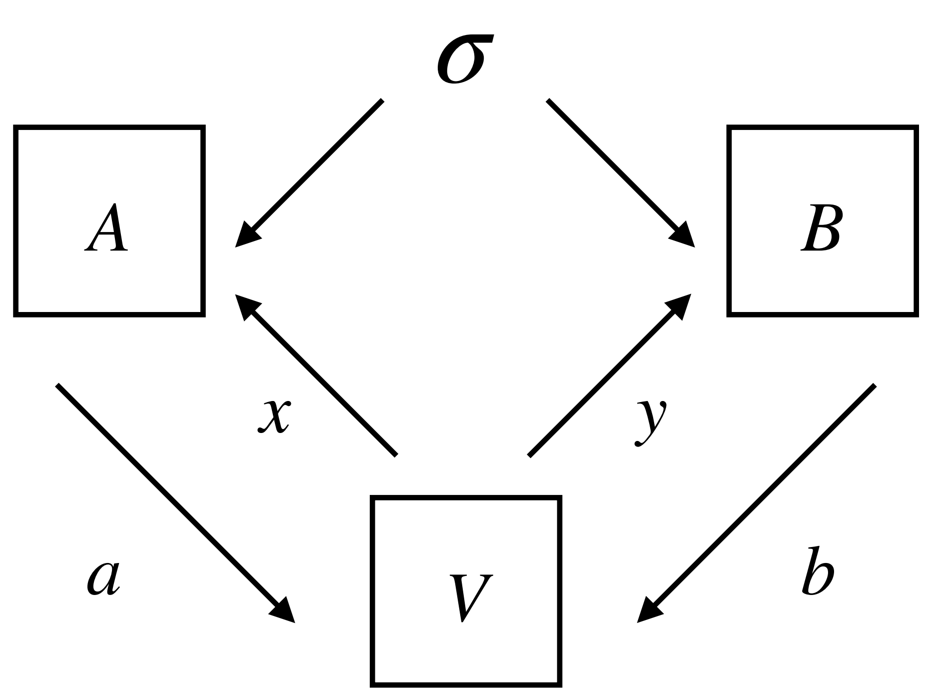
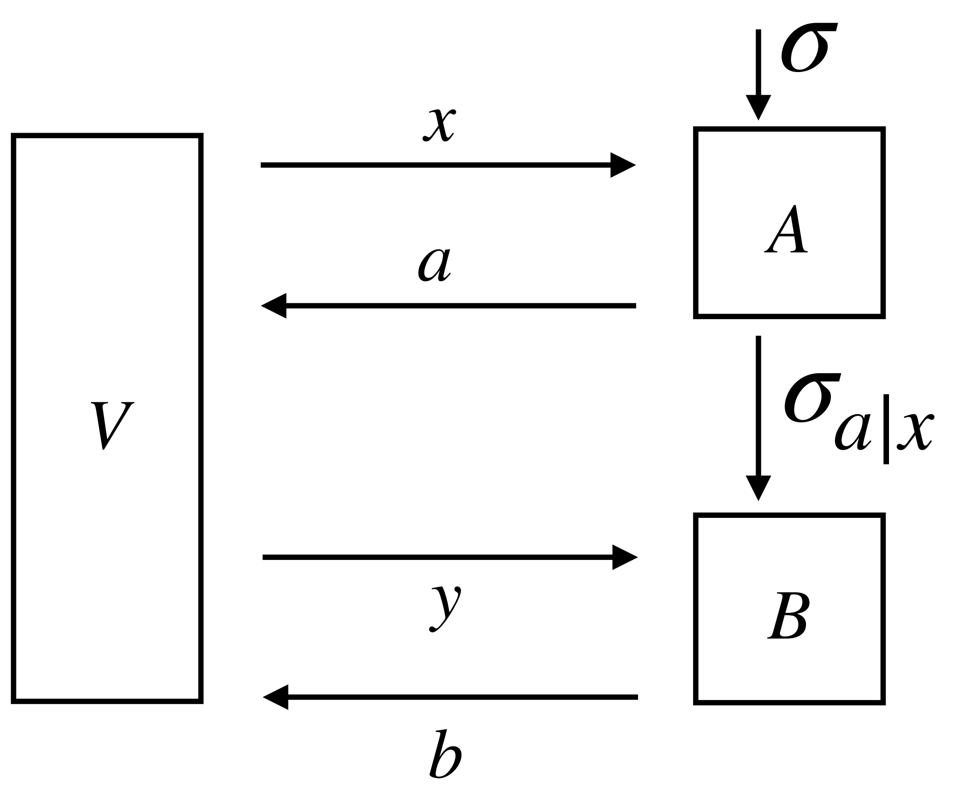
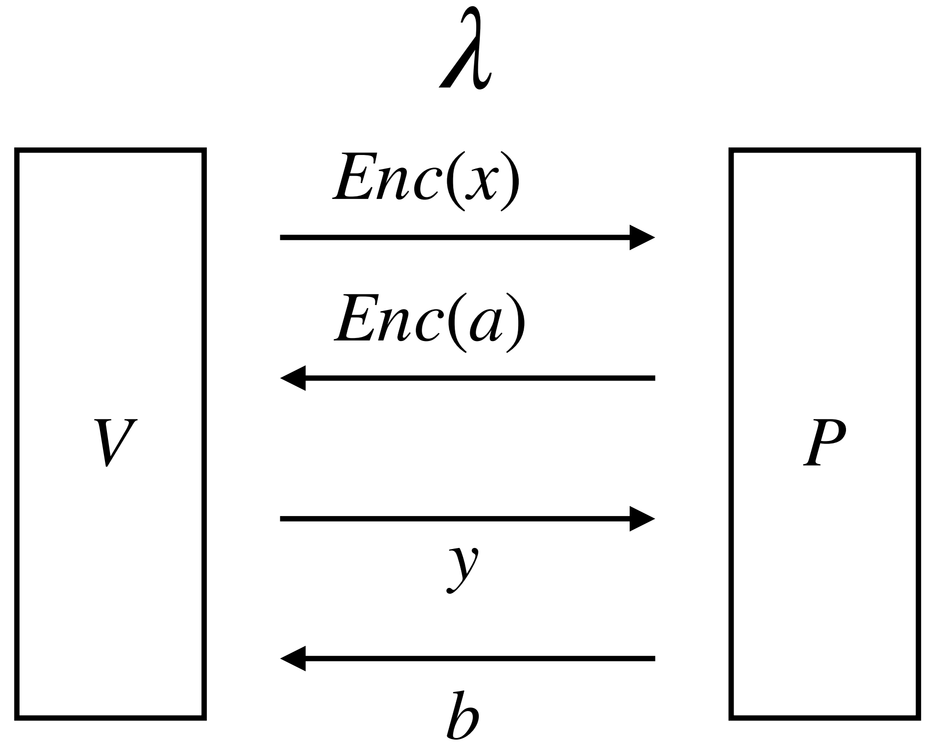
Nonlocal to sequential to compiled







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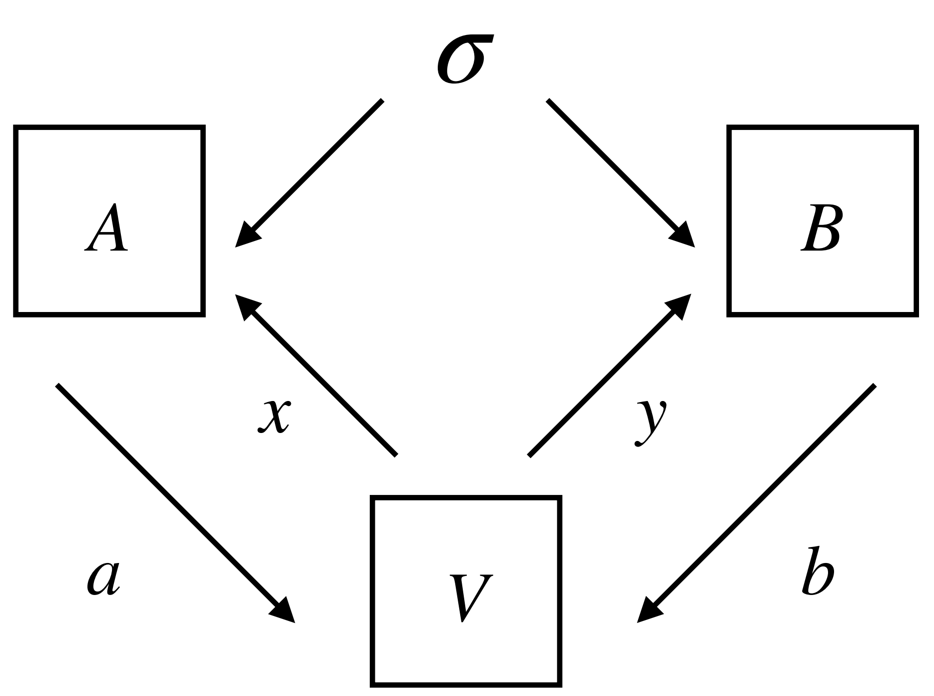
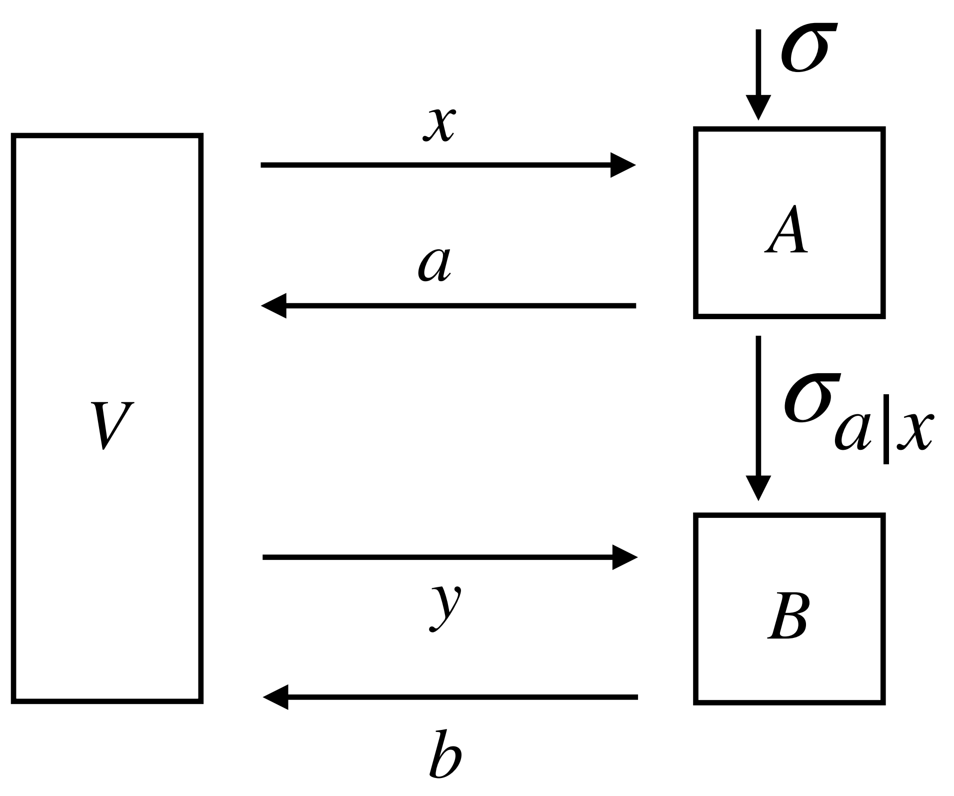
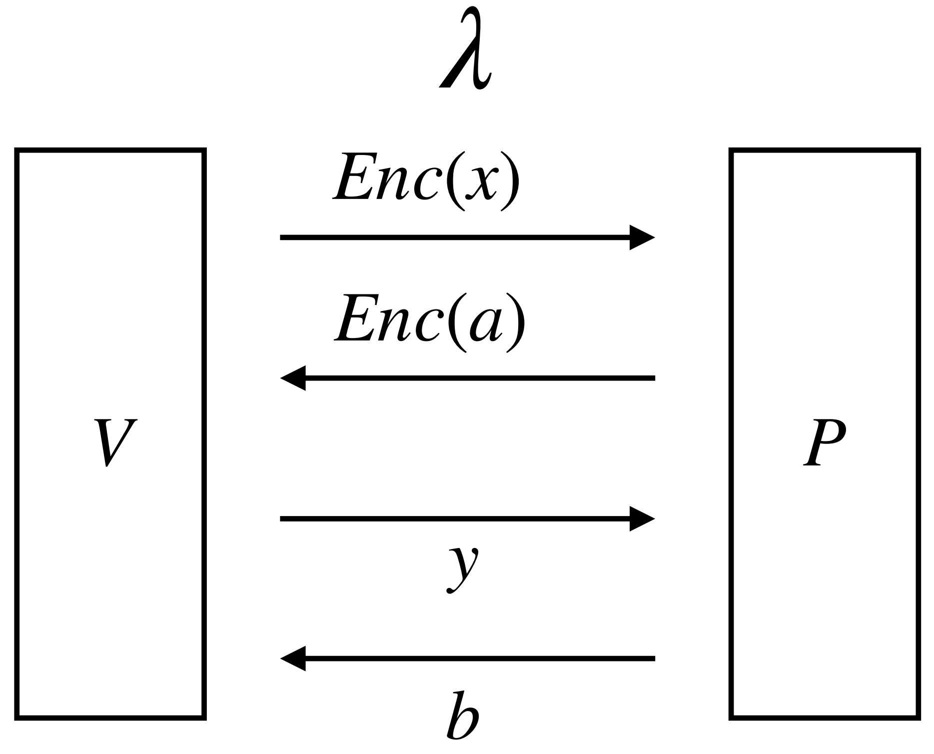
Nonlocal to sequential to compiled







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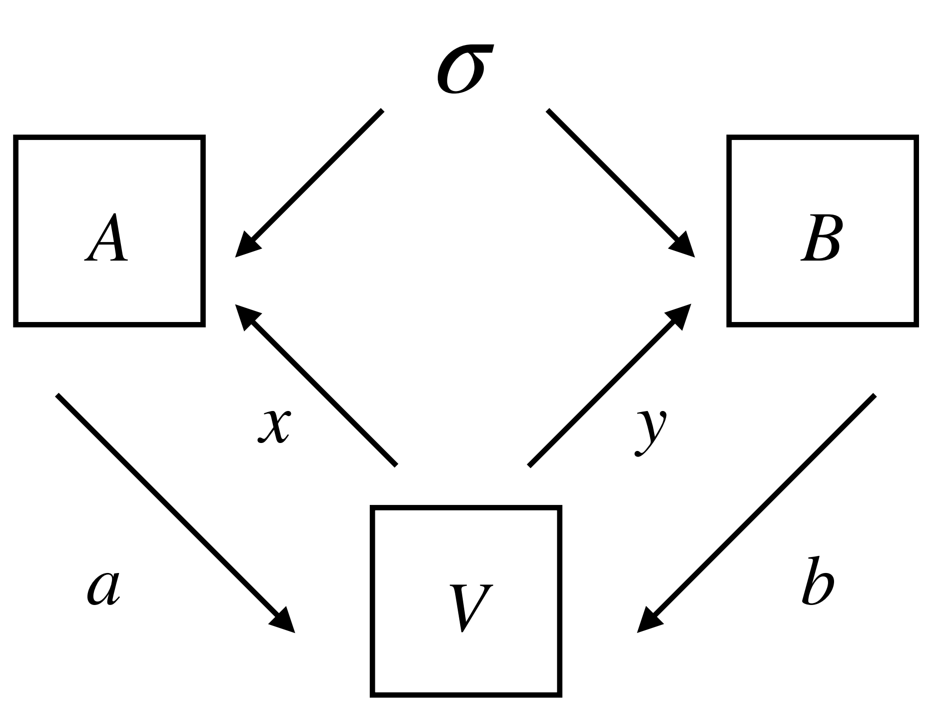
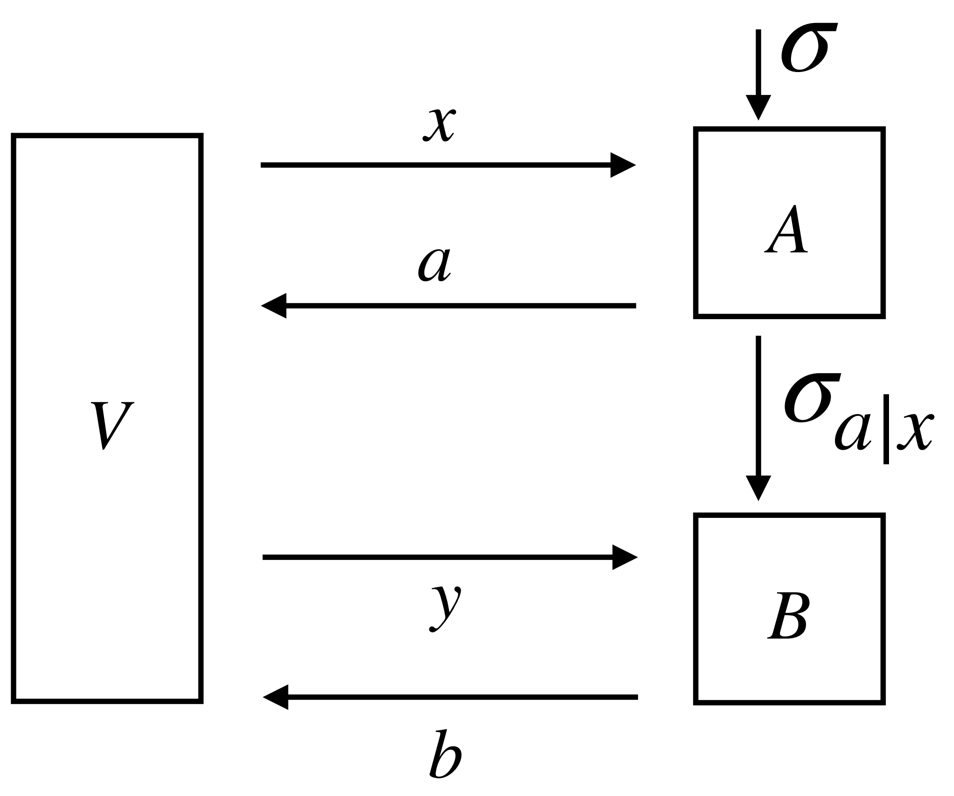
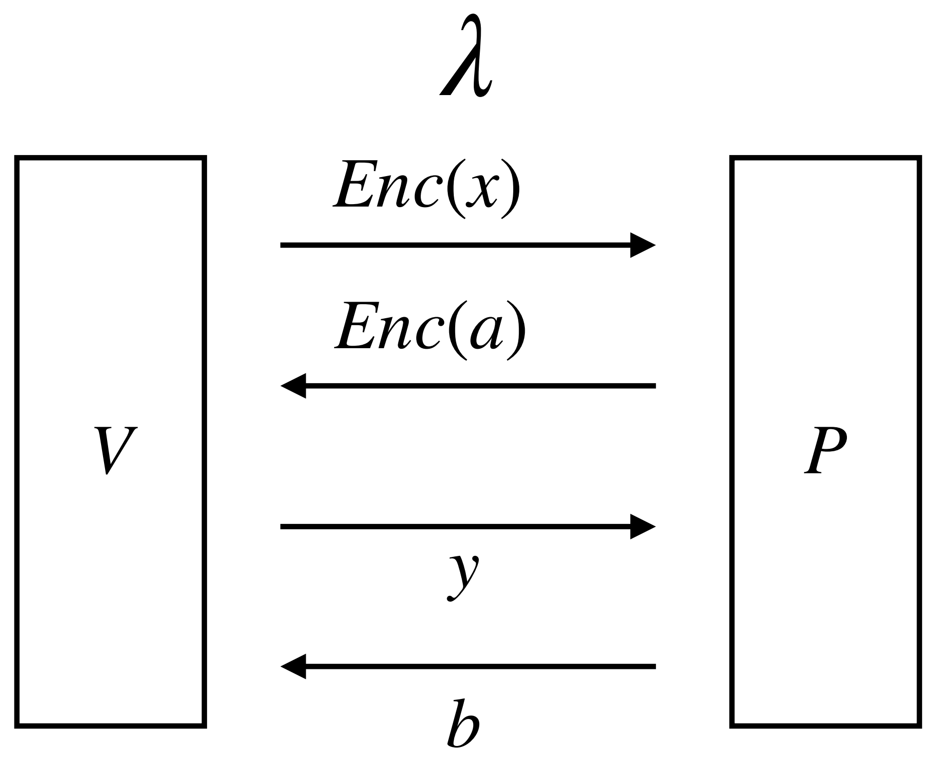
- Alice first measures and send the post-measured state $\sigma_{a|x}$
- $p(ab\,|\,xy) = \sigma_{a|x}(B_{b|y})$
- Strongly no-signaling: $|\sum_a p(ab\,|\,xy) - \sum_a p(ab\,|\,x'y)| = 0$

Nonlocal to sequential to compiled







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- Alice first measures and send the post-measured state $\sigma_{a|x}$
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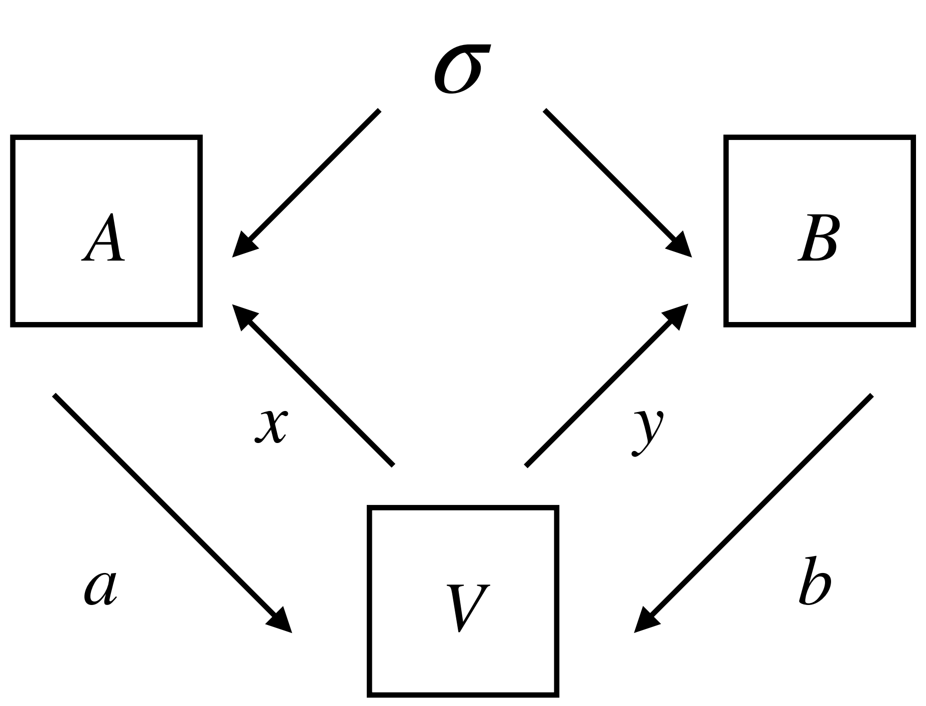
- Strongly no-signaling: $|\sum_a p(ab\,|\,xy) - \sum_a p(ab\,|\,x'y)\,| = 0$

Particularly $\sum_a \sigma_{a|x} = \sigma$

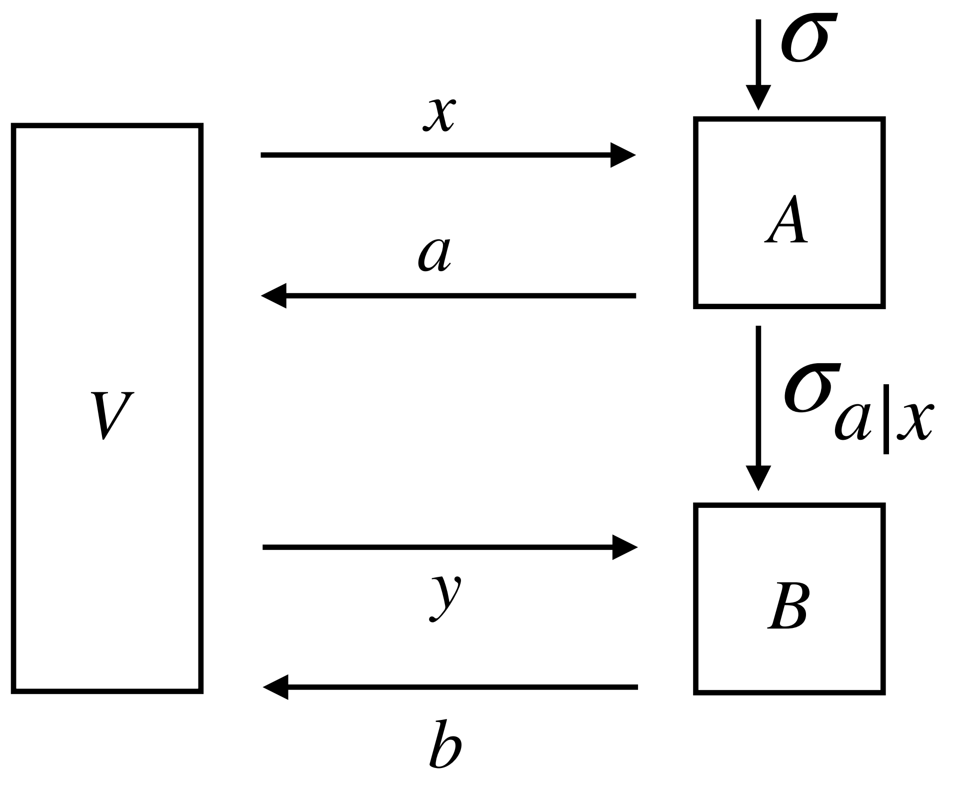
For P polynomial of any degrees

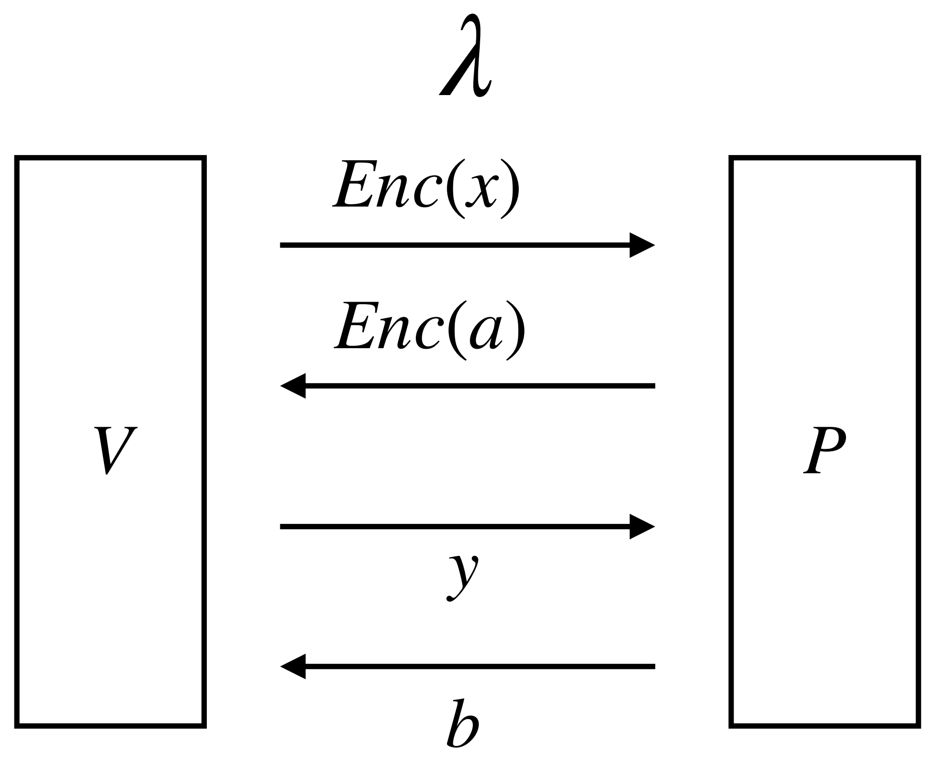
$$|\sum_a \sigma_{a|x}(P(\{B_{b|y}\})) - \sum_a \sigma_{a|x'}(P(\{B_{b|y}\}))| = 0$$

Nonlocal to sequential to compiled



$$\sigma_{a|x}(\,\cdot\,) = \sigma(A_{a|x}\cdot)$$





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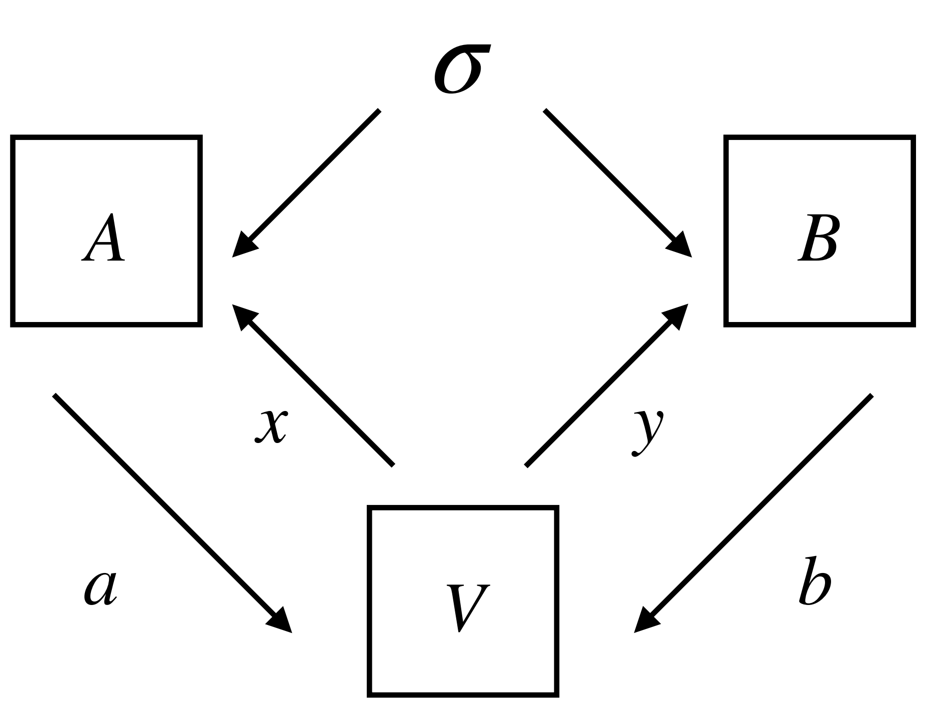
Nonlocal to sequential to compiled



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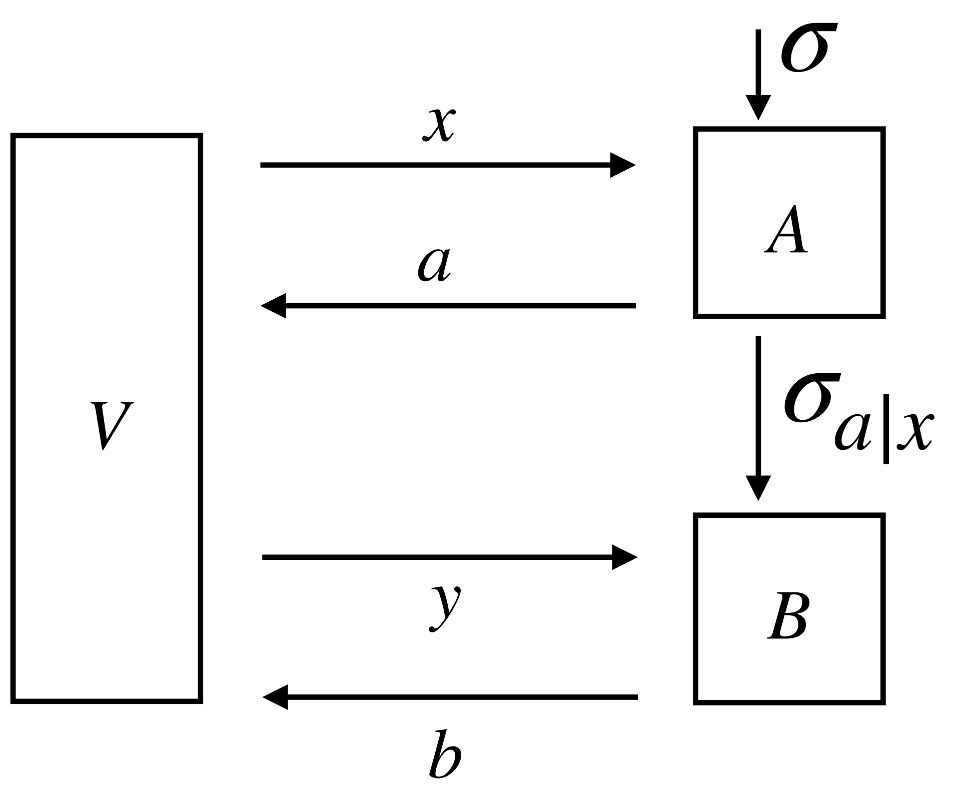

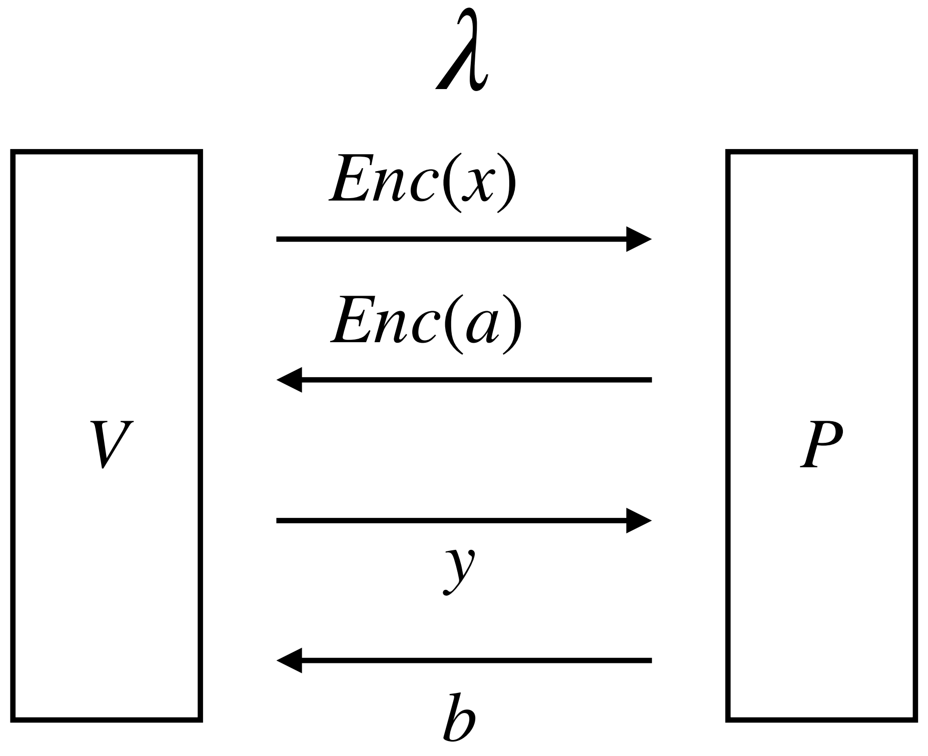

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- $\sigma_{a|x}$ that are strongly no-signaling:
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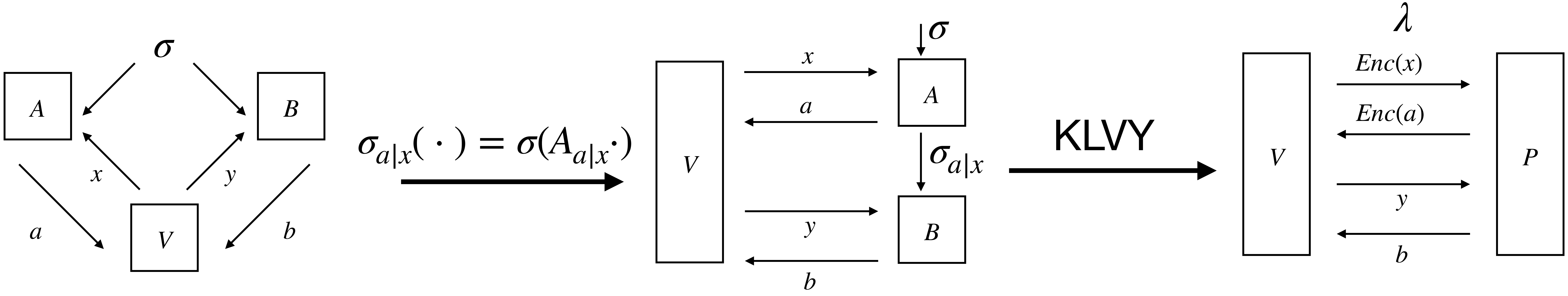
Nonlocal to sequential to compiled



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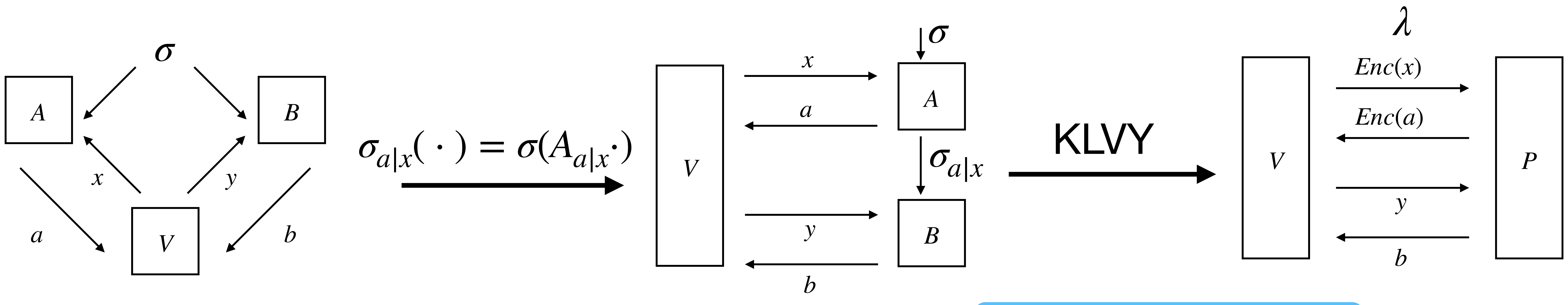


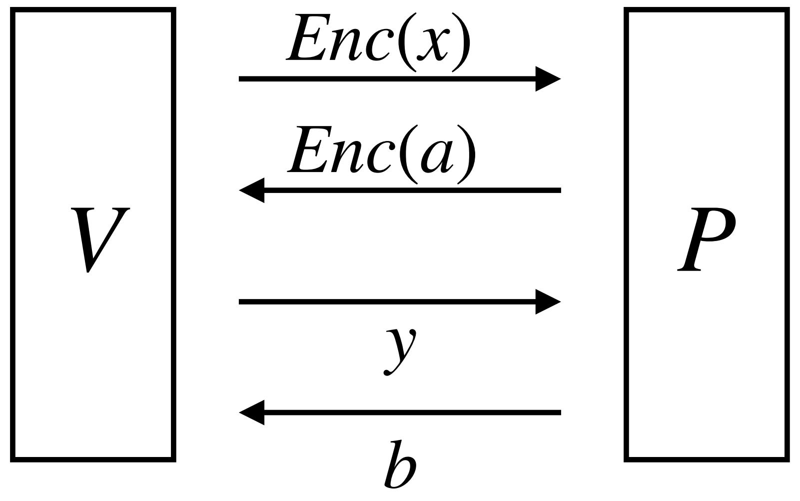
Converse direction?

- $\sum_a A_{a|x} = 1$
- $p(ab\,|\,xy) := \sigma(A_{a|x}B_{b|y})$
- $p(ab\,|\,xy) = \sigma_{a|x}(B_{b|y})$
- $\sigma_{a|x}$ that are strongly no-signaling:
  $|\sum_a \sigma_{a|x}(P(\{B_{b|y}\})) - \sum_a \sigma_{a|x'}(P(\{B_{b|y}\}))| = 0$

KMPSW24: compiled to sequential

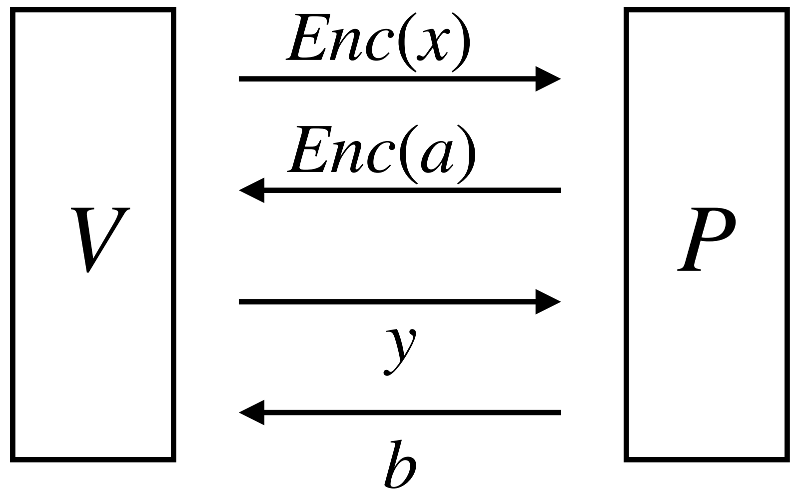
S_λ

$p^\lambda(ab|xy)$



KMPSW24: compiled to sequential

S_λ
$p^\lambda(ab \mid xy)$

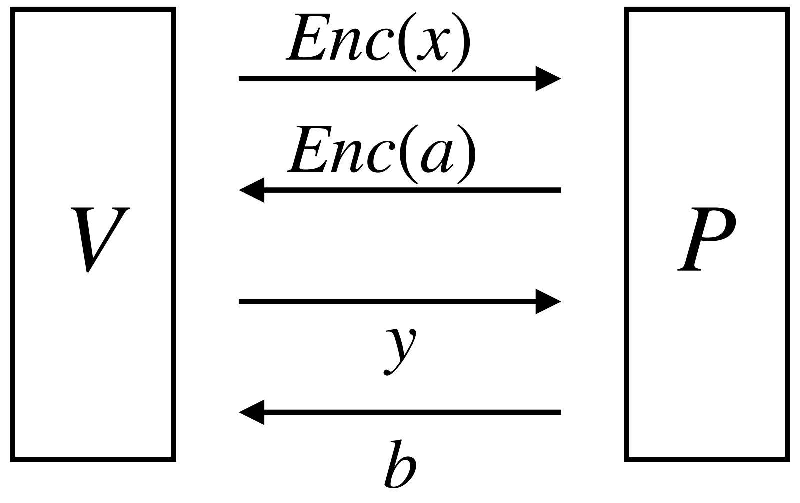


- Given compiled strategy S_λ correlation $p^\lambda(ab \mid xy)$

KMPSW24: compiled to sequential

S_λ

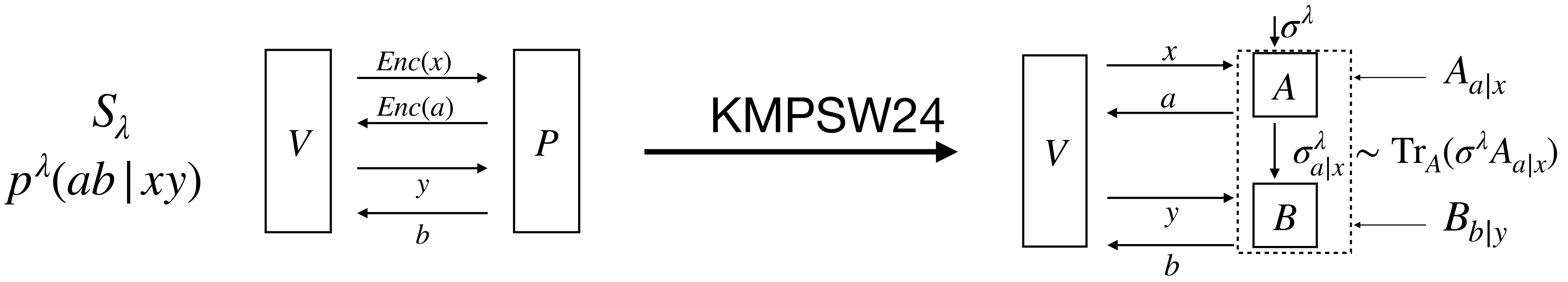
$p^\lambda(ab|xy)$



$\xrightarrow{\text{KMPSW24}}$

- Given compiled strategy S_λ correlation $p^\lambda(ab|xy)$

KMPSW24: compiled to sequential



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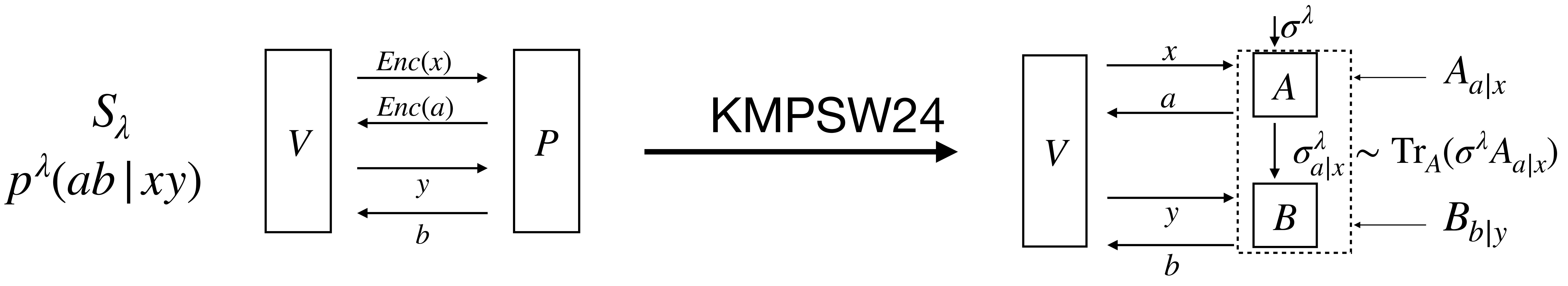
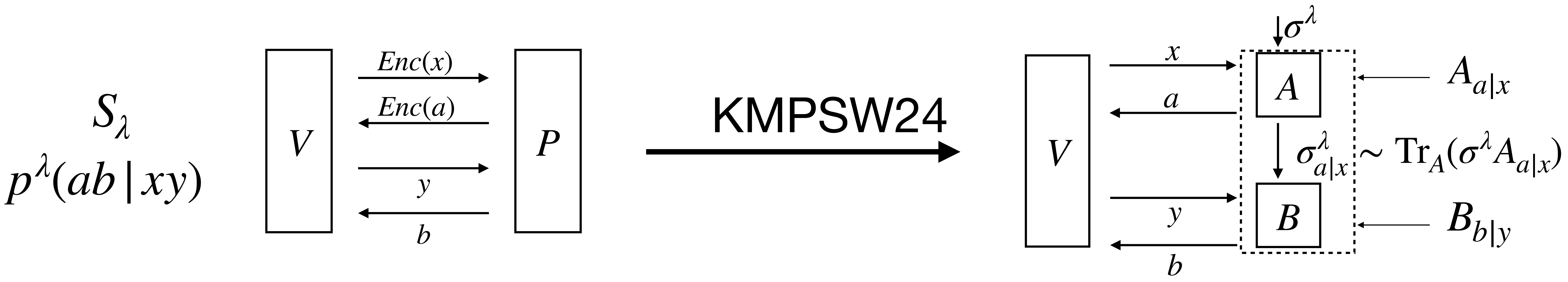
KMPSW24: compiled to sequential



- Given compiled strategy S_λ correlation $p^\lambda(ab|xy)$
- There exist encrypted post-measured states $\sigma^\lambda_{a|x}$

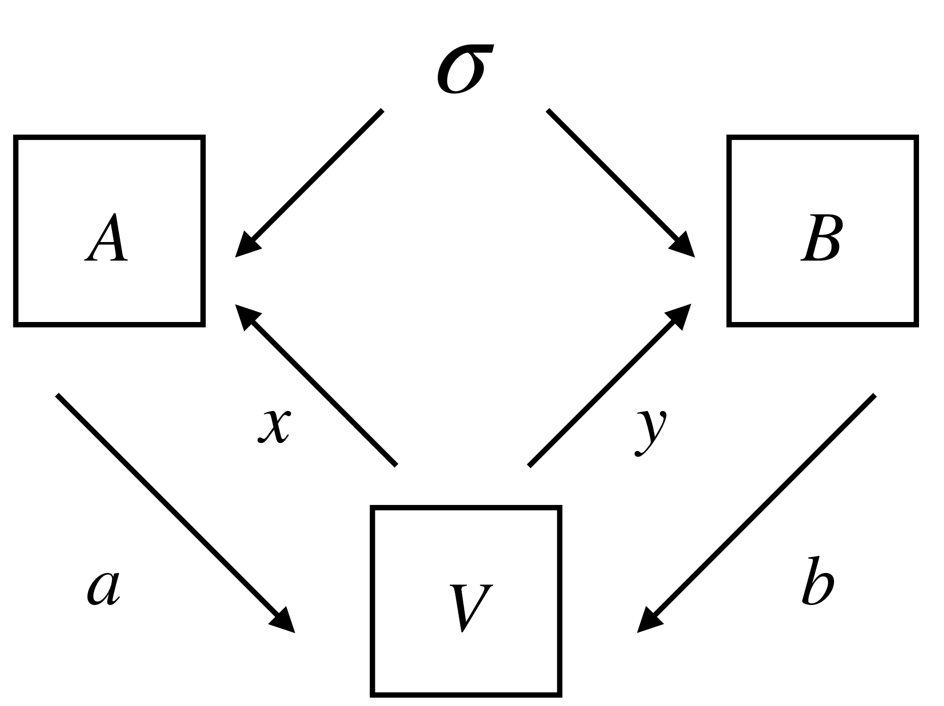
KMPSW24: compiled to sequential

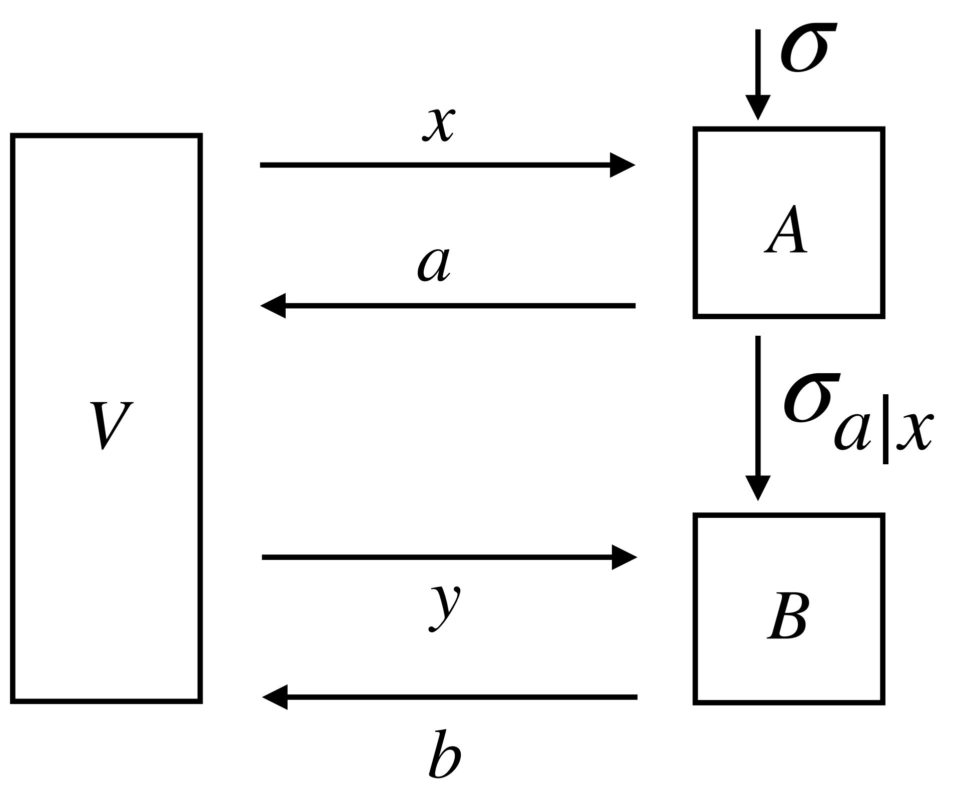


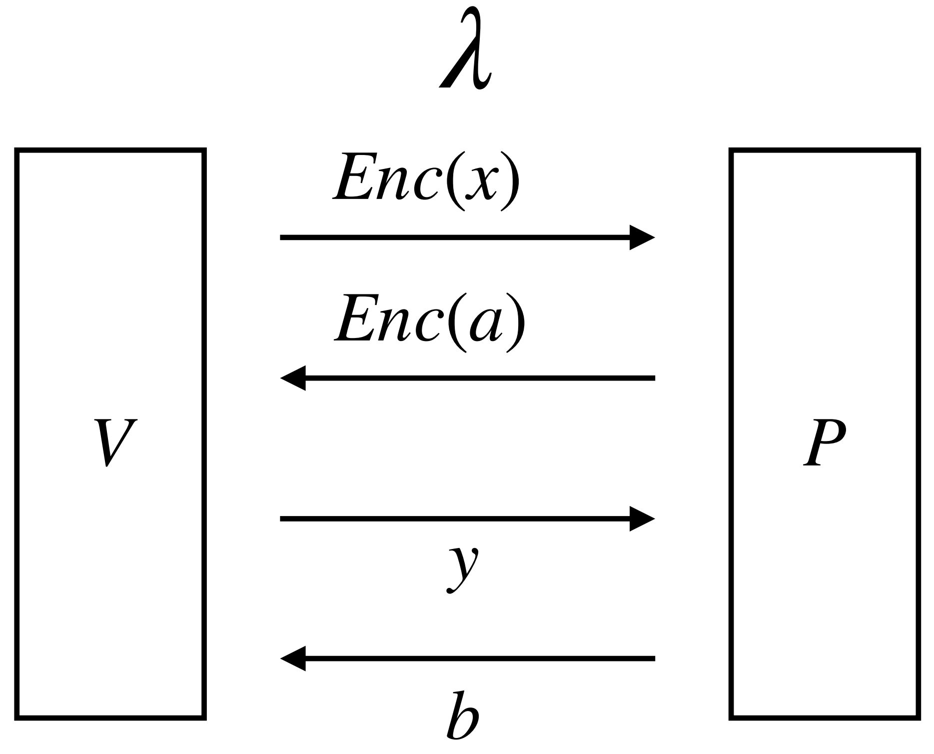
- Given compiled strategy S_λ correlation $p^\lambda(ab|xy)$
- There exist encrypted post-measured states $\sigma^\lambda_{a|x}$
- IND-CPA security/weakly no-signalling:

$$\left|\sum_a \sigma^\lambda_{a|x}(P(\{B_{b|y}\})) - \sum_a \sigma^\lambda_{a|x'}(P(\{B_{b|y}\}))\right| \leq \mathrm{negl}_P(\lambda)$$

KMPSW24: compiled to sequential to nonlocal







- $p(ab\,|\,xy) = \sigma_{a|x}(B_{b|y})$
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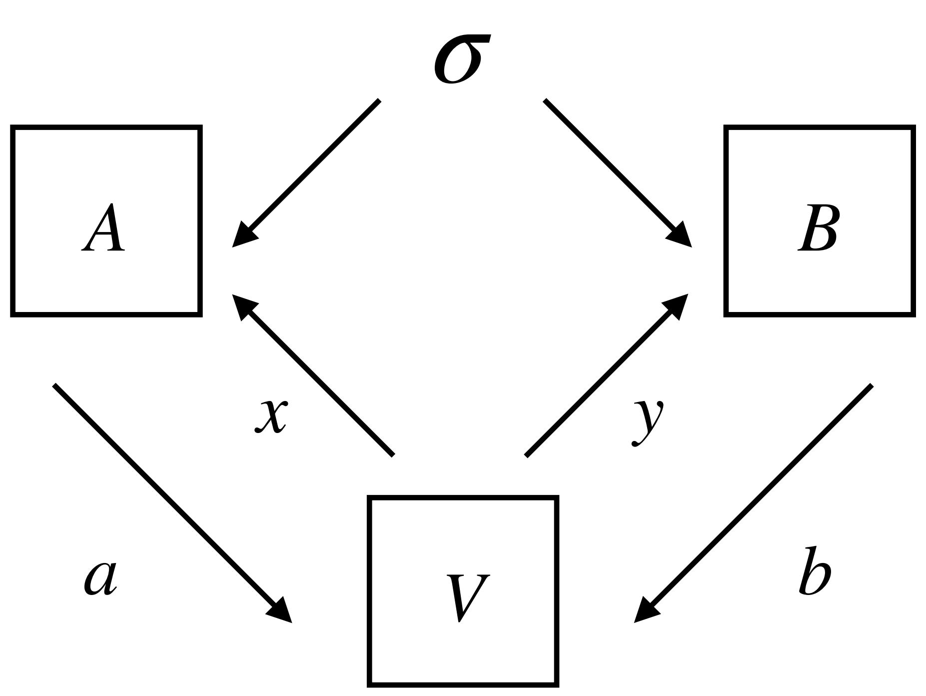
- $\sigma_{a|x}$ that are strongly no-signaling: $\left|\sum_a \sigma_{a|x}(P(\{B_{b|y}\})) - \sum_a \sigma_{a|x'}(P(\{B_{b|y}\}))\right| = 0$
- $p(ab\,|\,xy) := \sigma(A_{a|x}B_{b|y})$

KMPSW24: compiled to sequential to nonlocal



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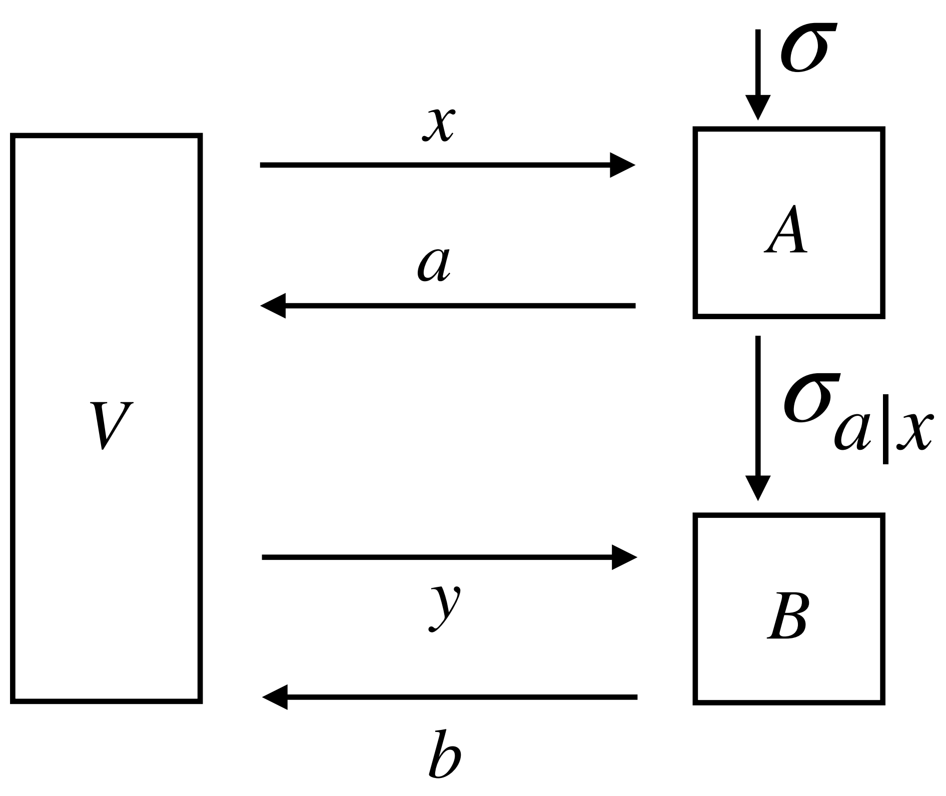

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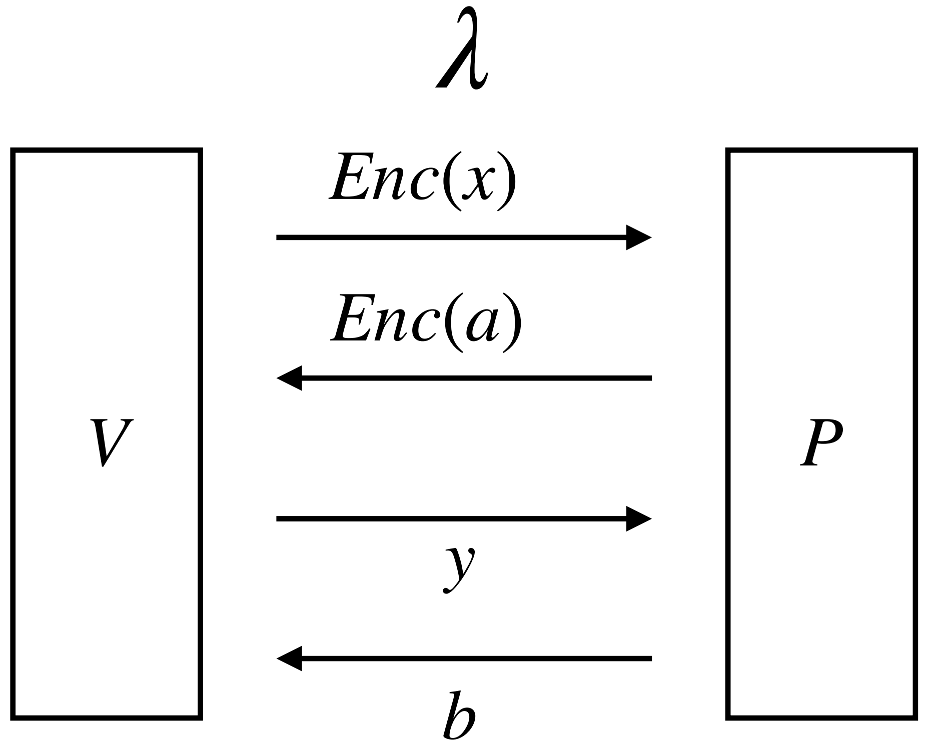
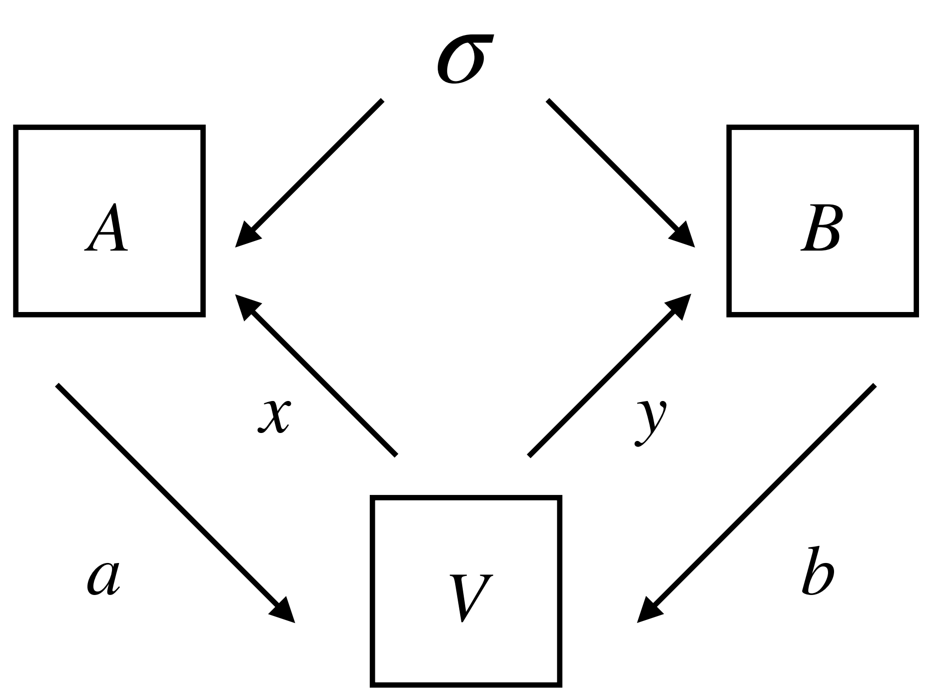


- Encrypted post-measured state $\sigma^{\lambda}_{a|x}$

KMPSW24: compiled to sequential to nonlocal



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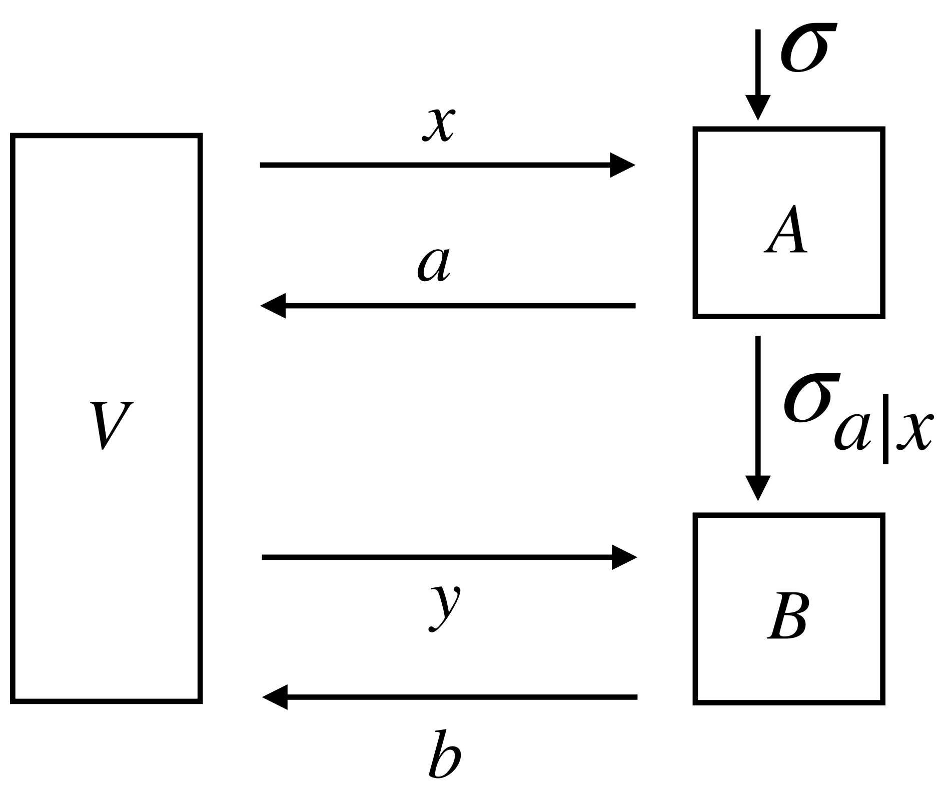

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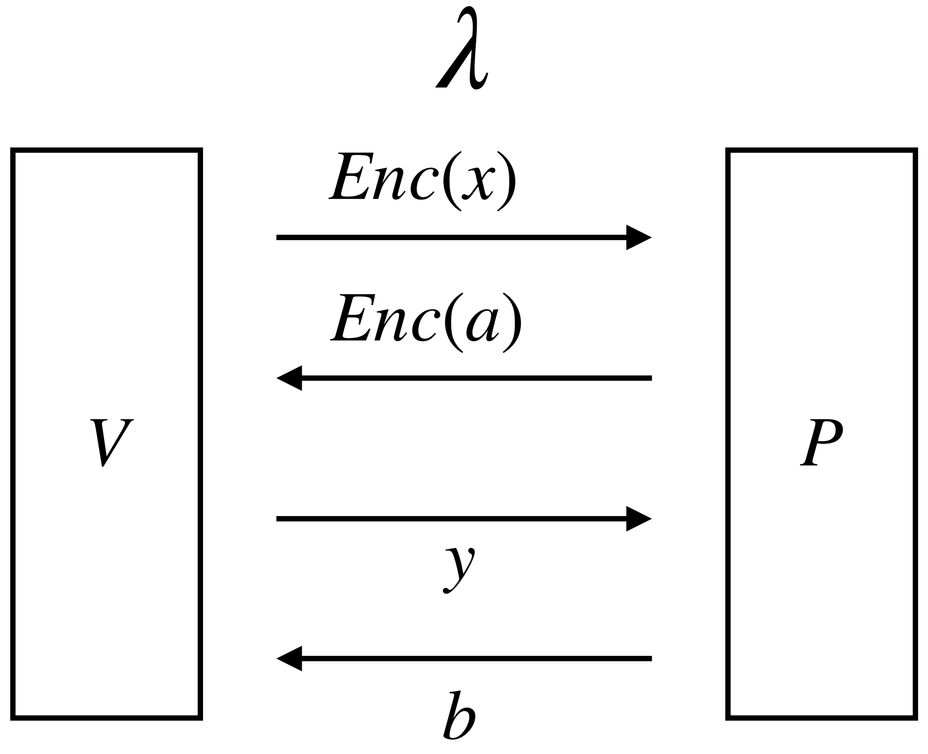
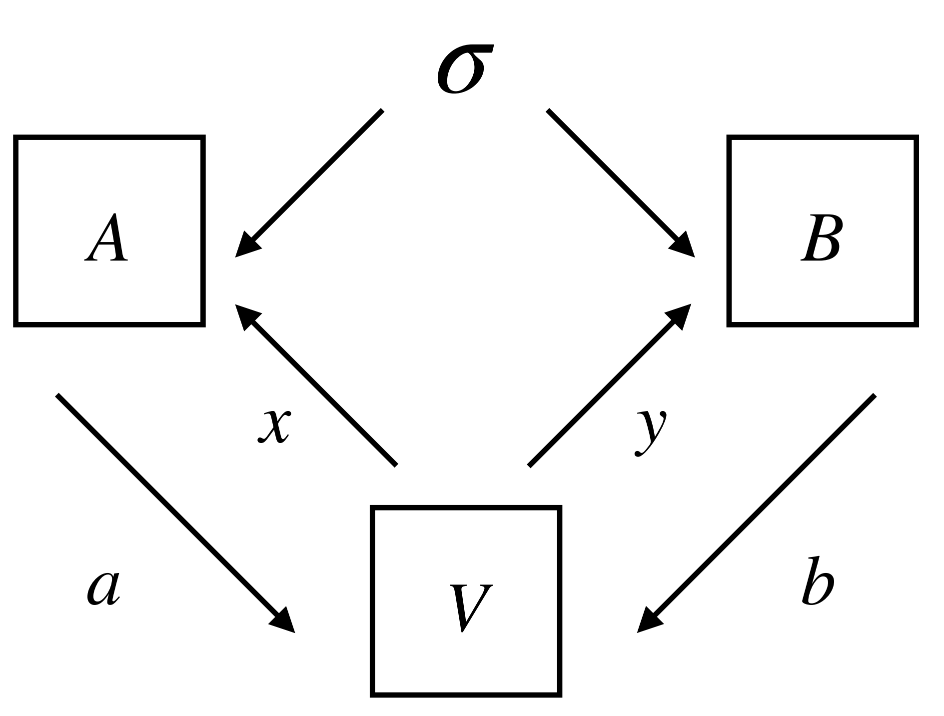


- Encrypted post-measured state $\sigma^{\lambda}_{a|x}$
- Weakly no-signaling:

KMPSW24: compiled to sequential to nonlocal



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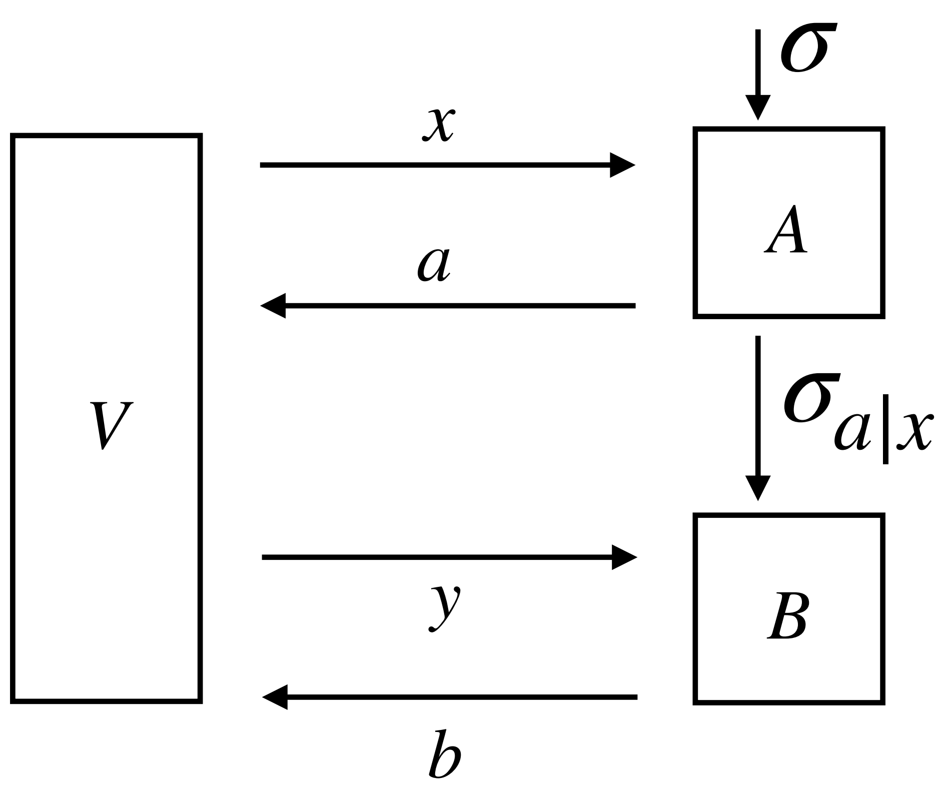

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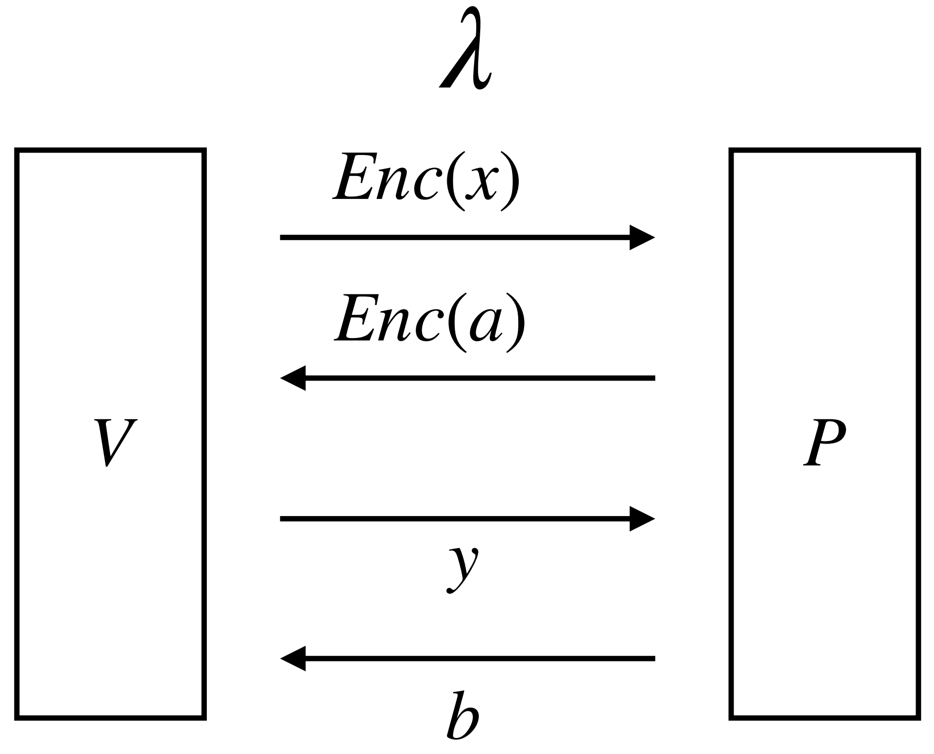

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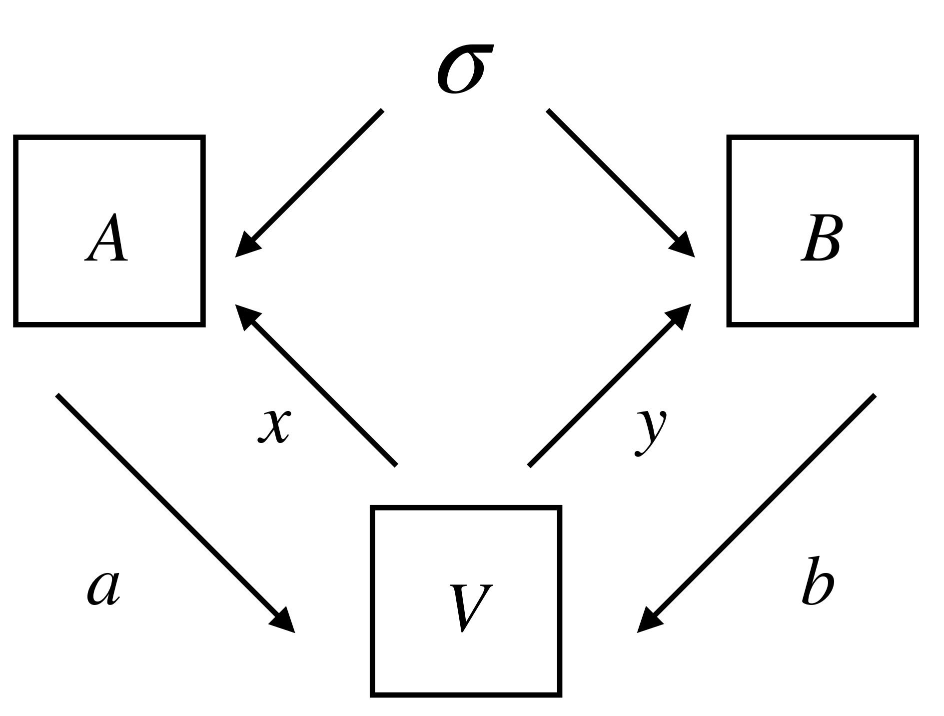
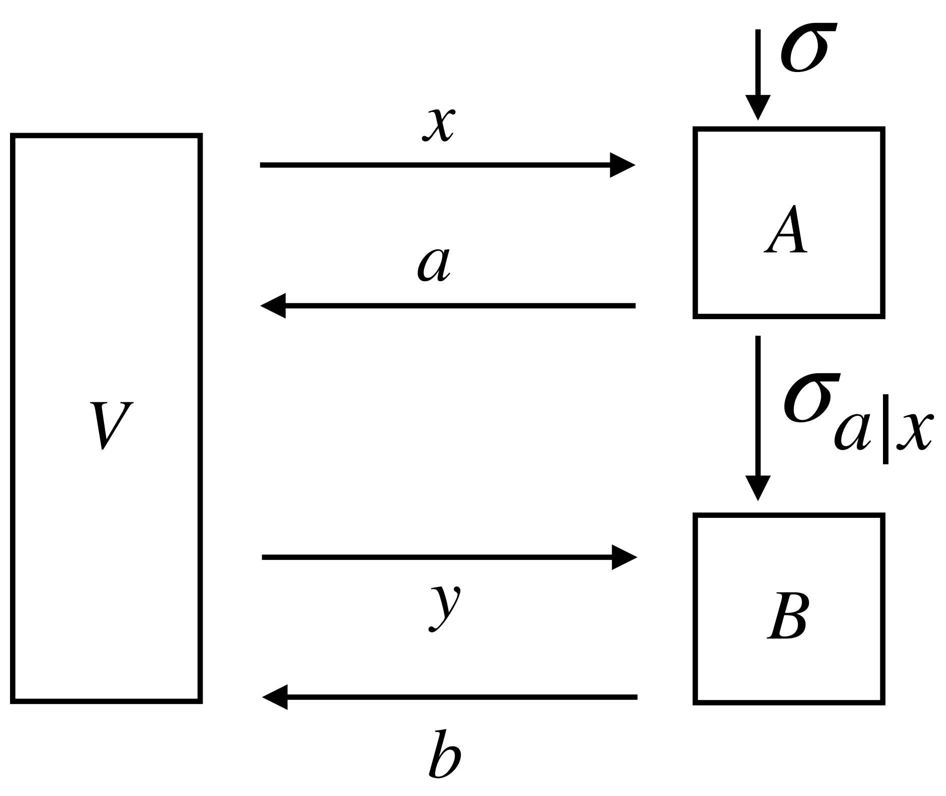
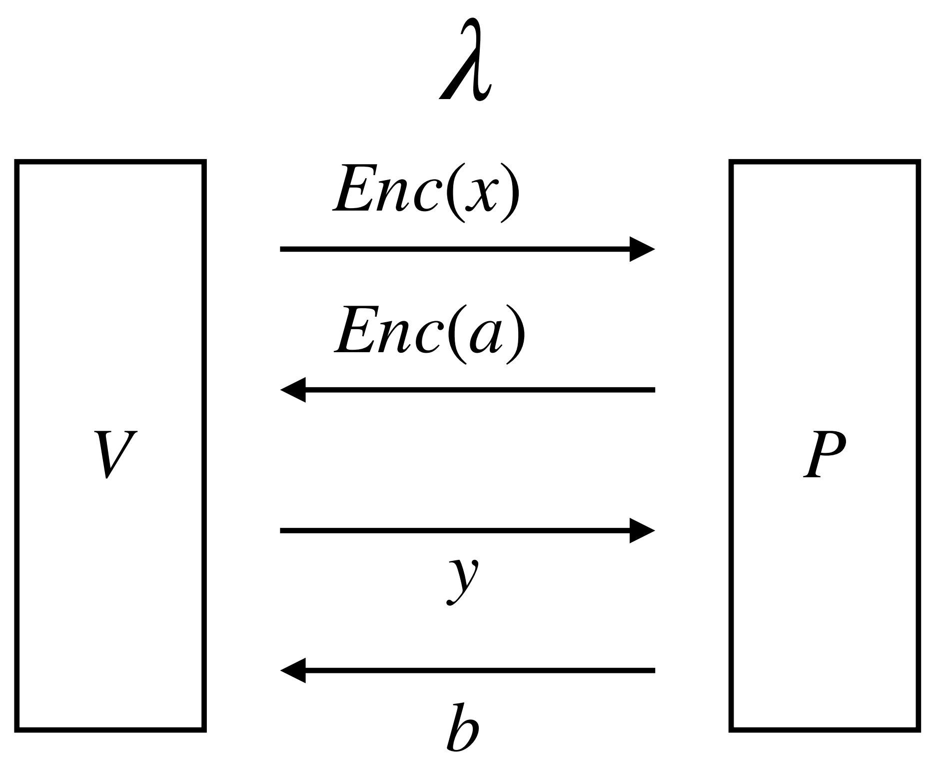
KMPSW24: compiled to sequential to nonlocal







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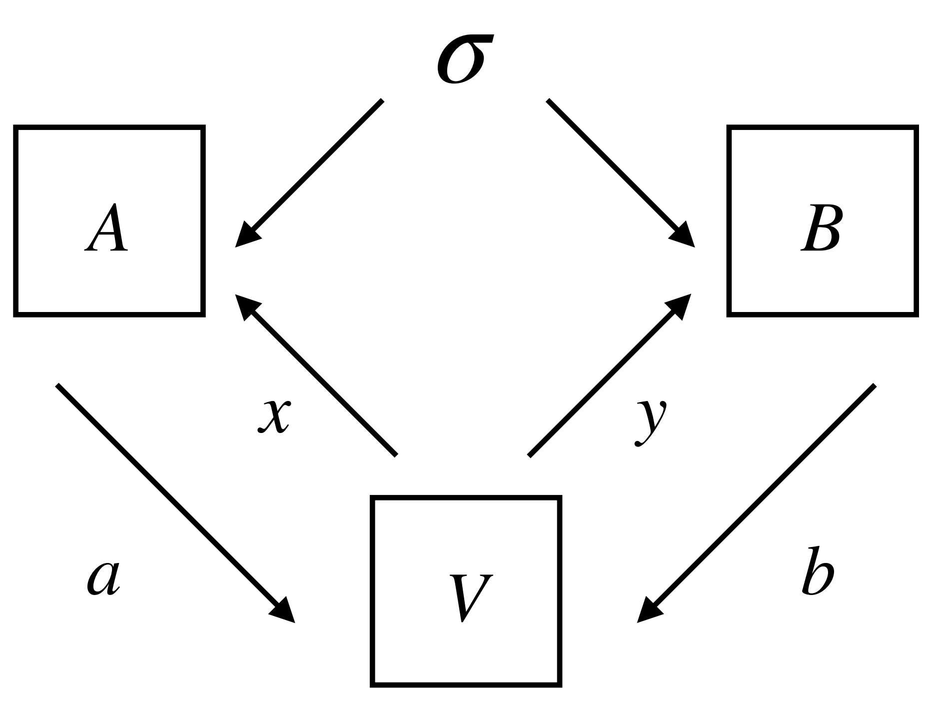
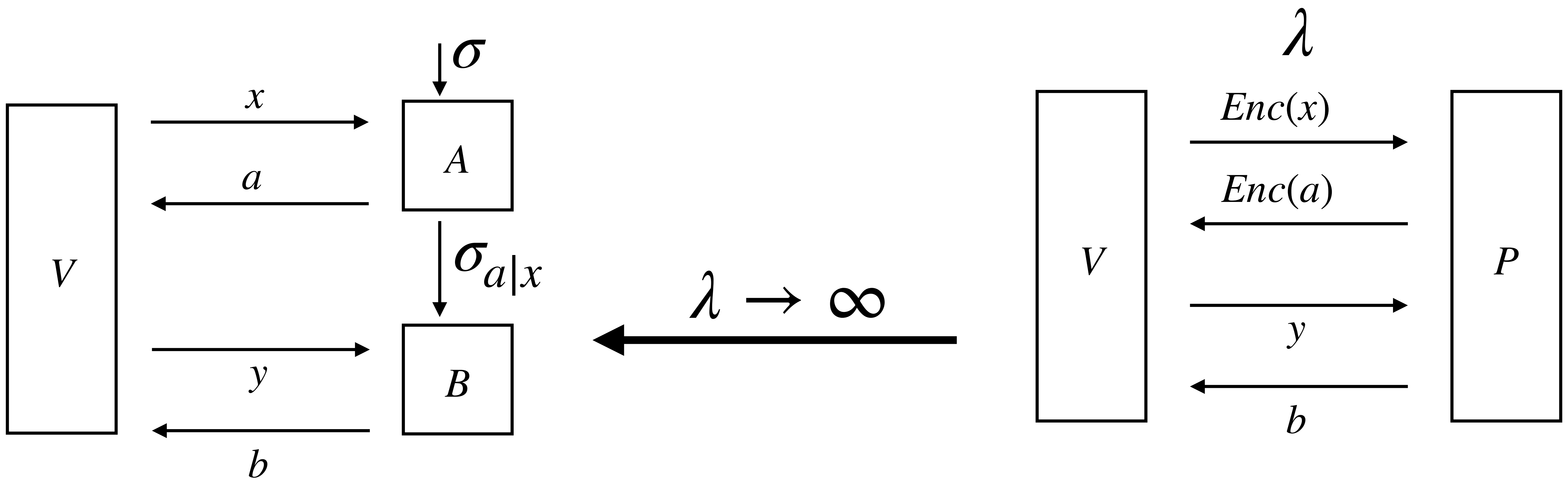
- Encrypted post-measured state $\sigma^{\lambda}_{a|x}$
- Weakly no-signaling:

Asymptotic $\lambda \to \infty$:
$|\sum_a \sigma_{a|x}(P(\{B_{b|y}\})) - \sum_a \sigma_{a|x'}(P(\{B_{b|y}\}))| \leq \mathrm{negl}_P(\lambda) \to 0$

KMPSW24: compiled to sequential to nonlocal





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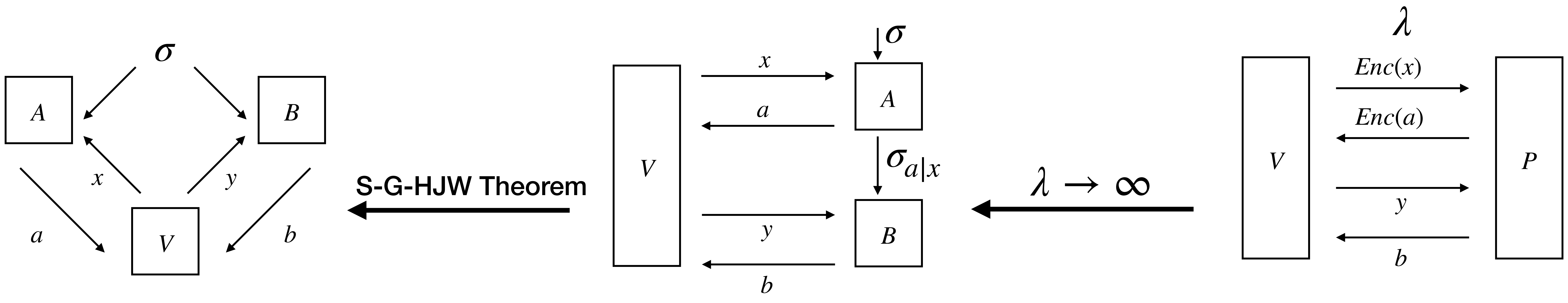
- Encrypted post-measured state $\sigma^{\lambda}_{a|x}$
- Weakly no-signaling:

Asymptotic $\lambda \to \infty$:

$$|\sum_a \sigma_{a|x}(P(\{B_{b|y}\})) - \sum_a \sigma_{a|x'}(P(\{B_{b|y}\}))| \leq \mathrm{negl}_P(\lambda) \to 0$$

KMPSW24: compiled to sequential to nonlocal



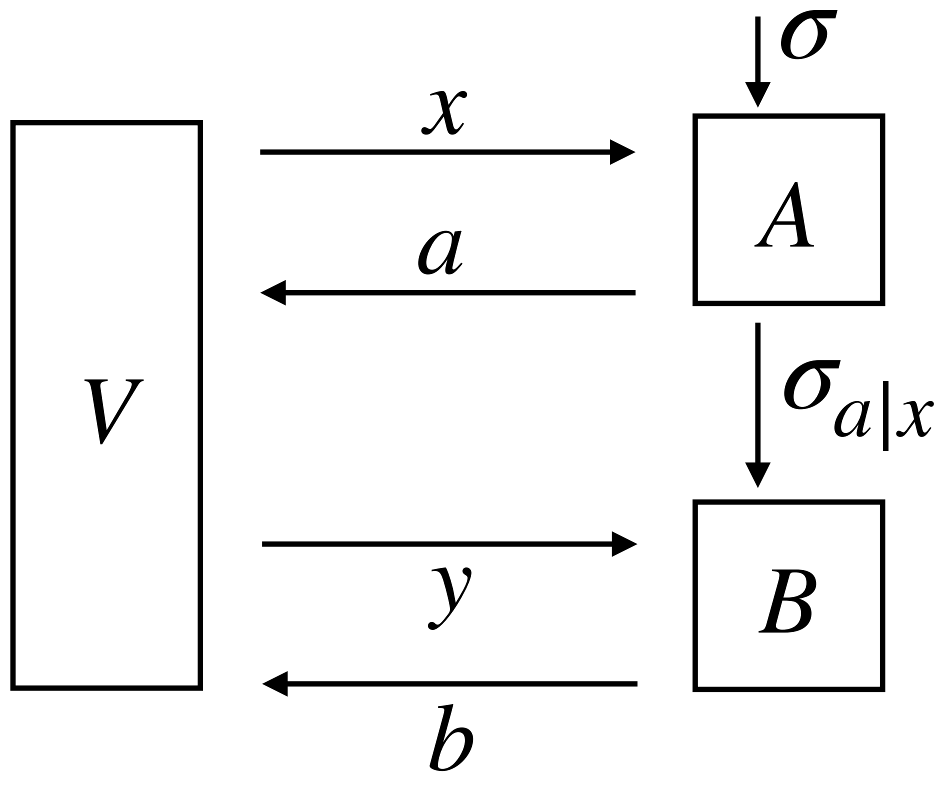
- $\sum_a A_{a|x} = 1$
- $p(ab\,|\,xy) := \sigma(A_{a|x}B_{b|y})$

- $p(ab\,|\,xy) = \sigma_{a|x}(B_{b|y})$
- $\sigma_{a|x}$ that are strongly no-signaling:
  $|\sum_a \sigma_{a|x}(P(\{B_{b|y}\})) - \sum_a \sigma_{a|x'}(P(\{B_{b|y}\}))| = 0$

- Encrypted post-measured state $\sigma^{\lambda}_{a|x}$
- Weakly no-signaling:

Asymptotic $\lambda \to \infty$:
$|\sum_a \sigma_{a|x}(P(\{B_{b|y}\})) - \sum_a \sigma_{a|x'}(P(\{B_{b|y}\}))| \leq \mathrm{negl}_P(\lambda) \to 0$

Sequential strategy to NPA hierarchy



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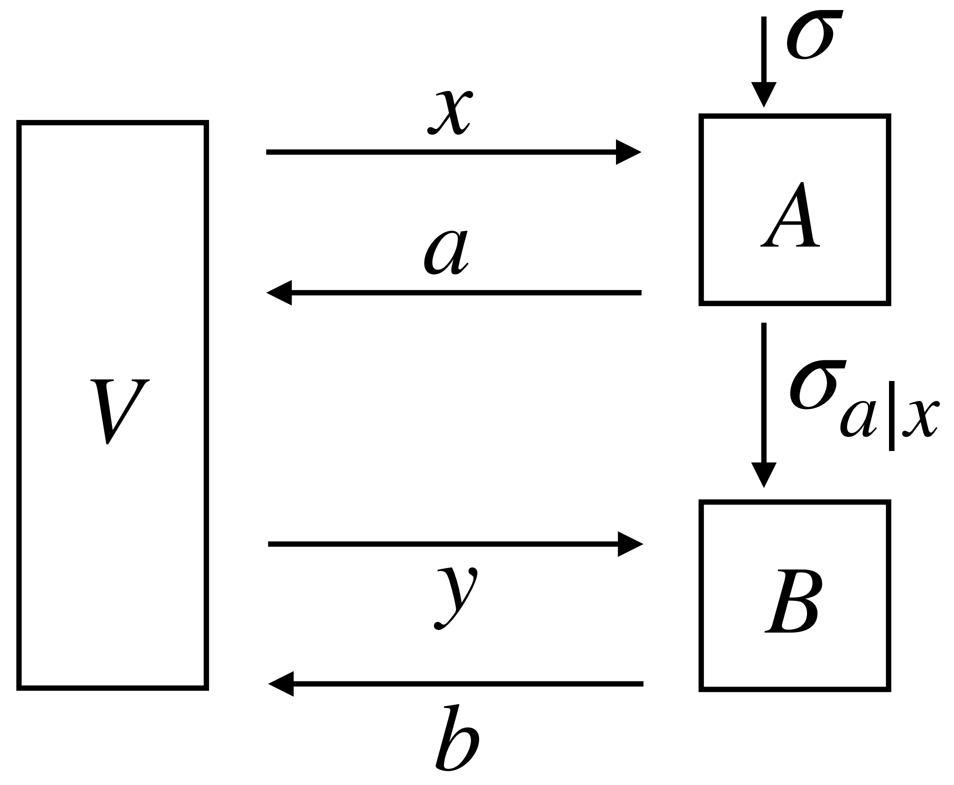

- Ideal sequential strategy: positive linear functionals $\sigma_{a|x}$ for all (x, a)

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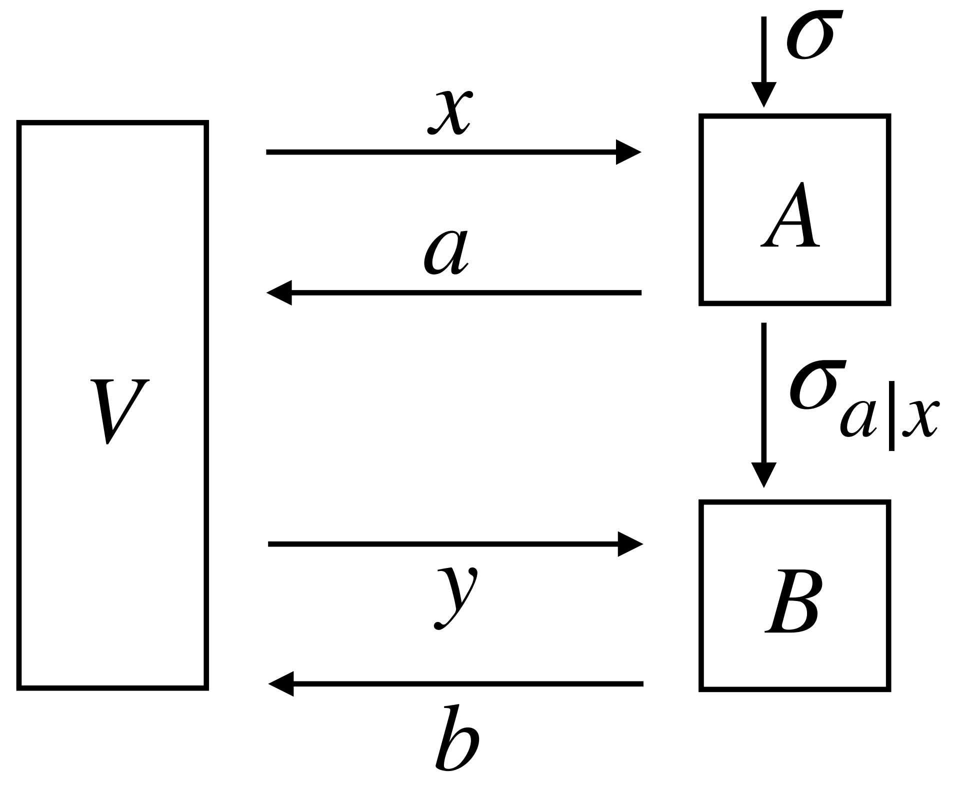

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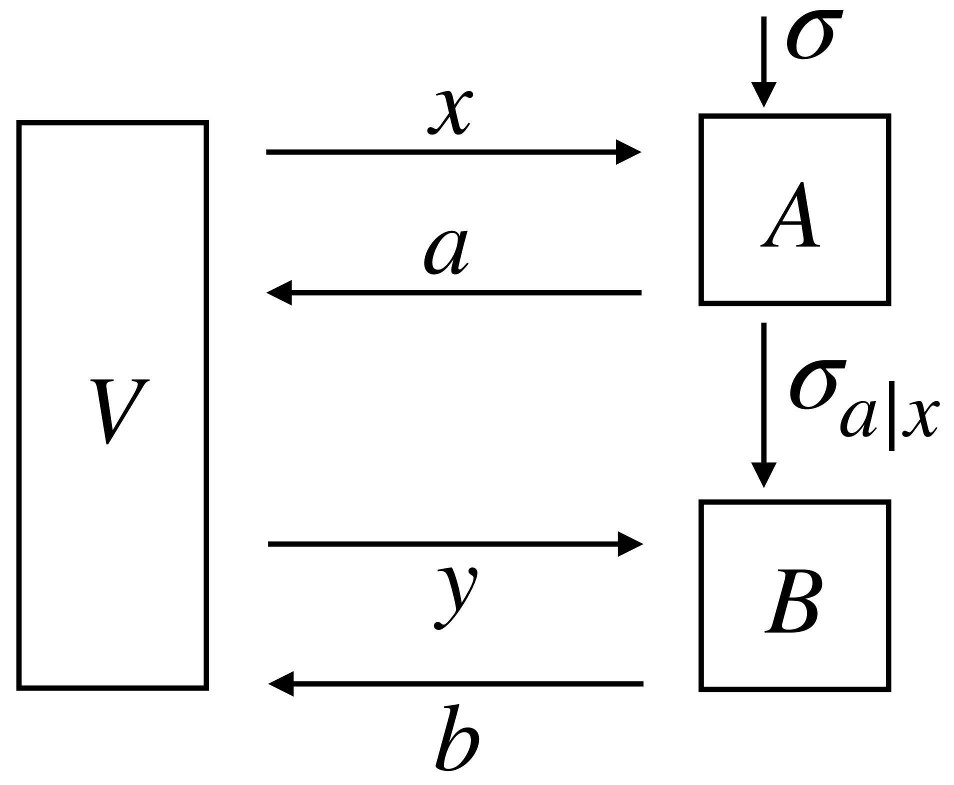

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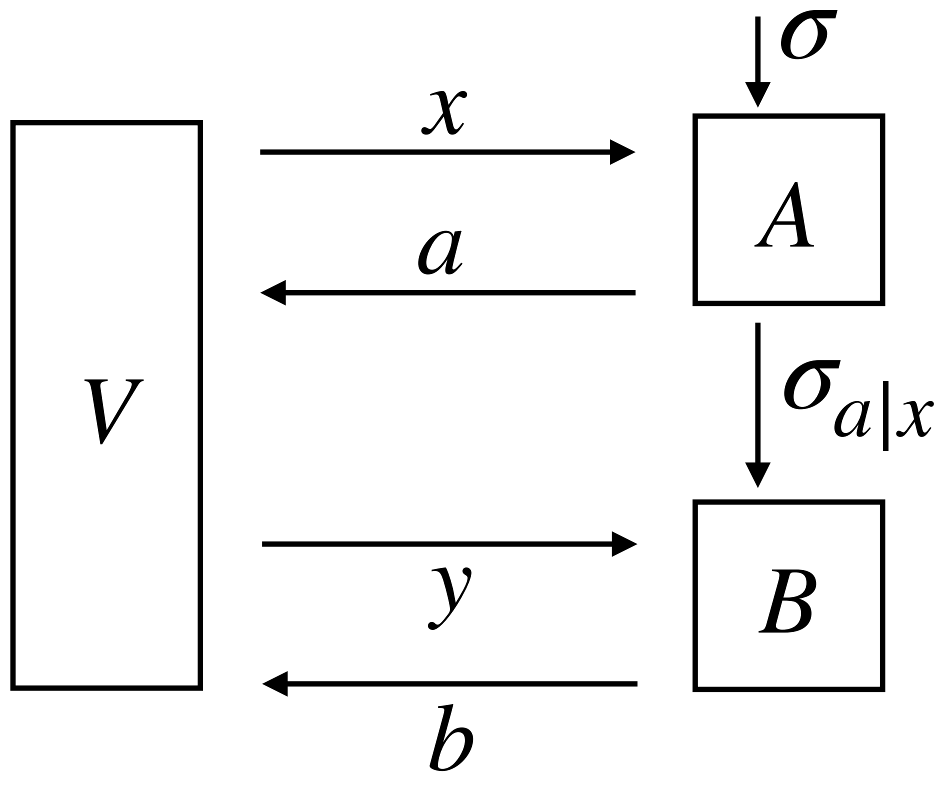

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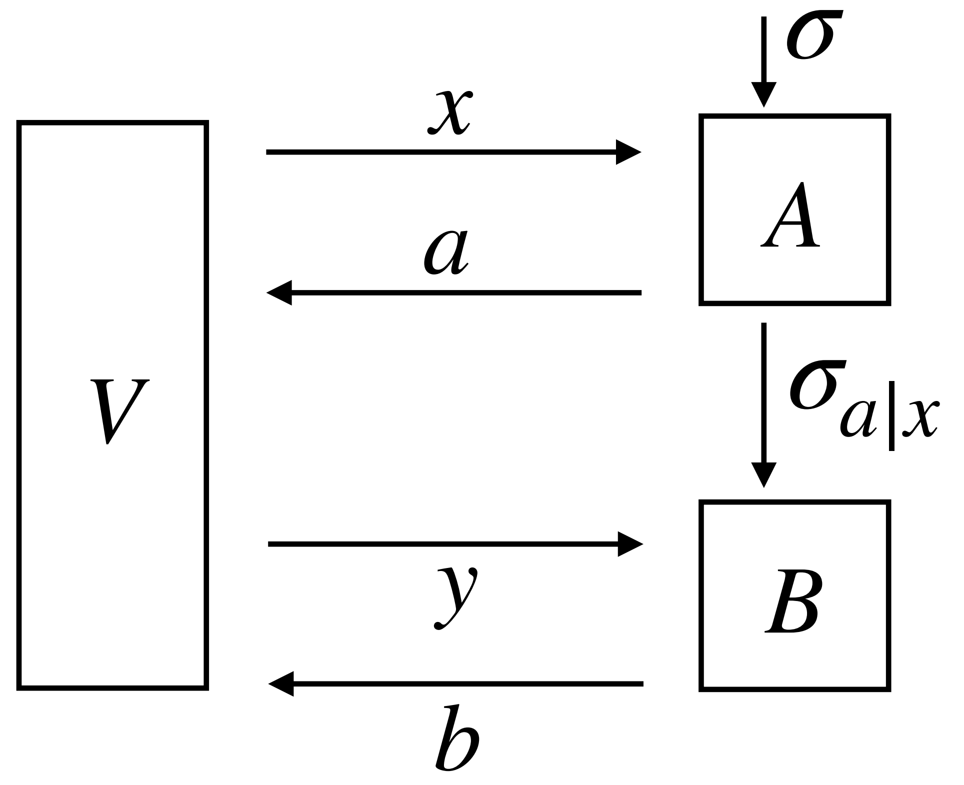

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- Maximization with P of all degrees is infeasible! Look at $\deg(P) \leq 2n$?

To n-th sequential NPA hierarchy

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- n-th level seqNPA: positive linear map $\sigma^n_{a|x}$ only for P of degree $\leq 2n$

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Buzzword: (PSD) moment matrix $\Gamma(a\,|\,x)$
$\Gamma(a\,|\,x)_{B_{b_1|y_1},B_{b_2|y_2}} = \sigma^n_{a|x}(B^*_{b_1|y_1}B_{b_2|y_2})$

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Sequential NPA hierarchy vs ideal strategy

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(n) positive linear map $\sigma^n_{a|x}$ for P of degree $\leq 2n$, max score

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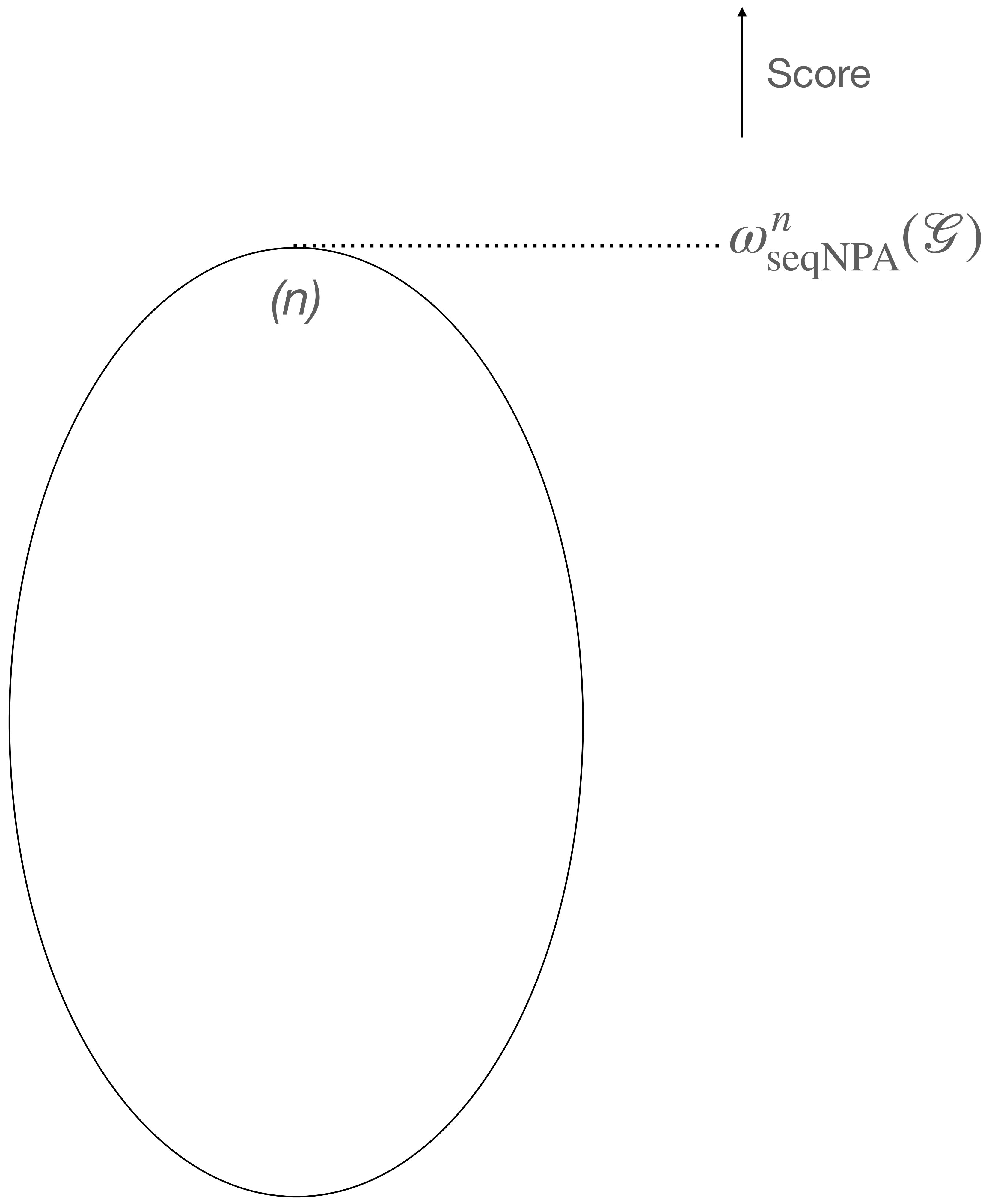

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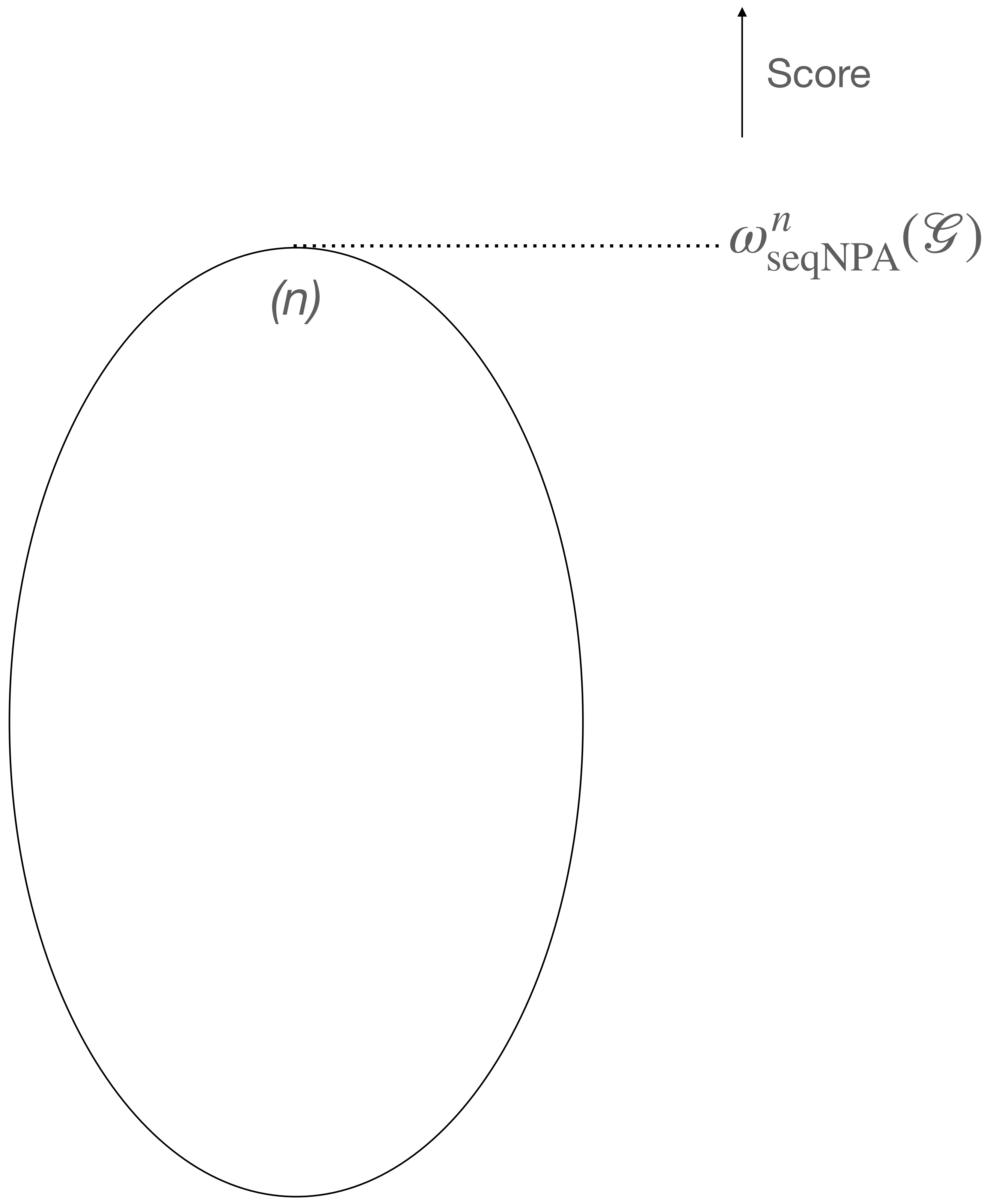

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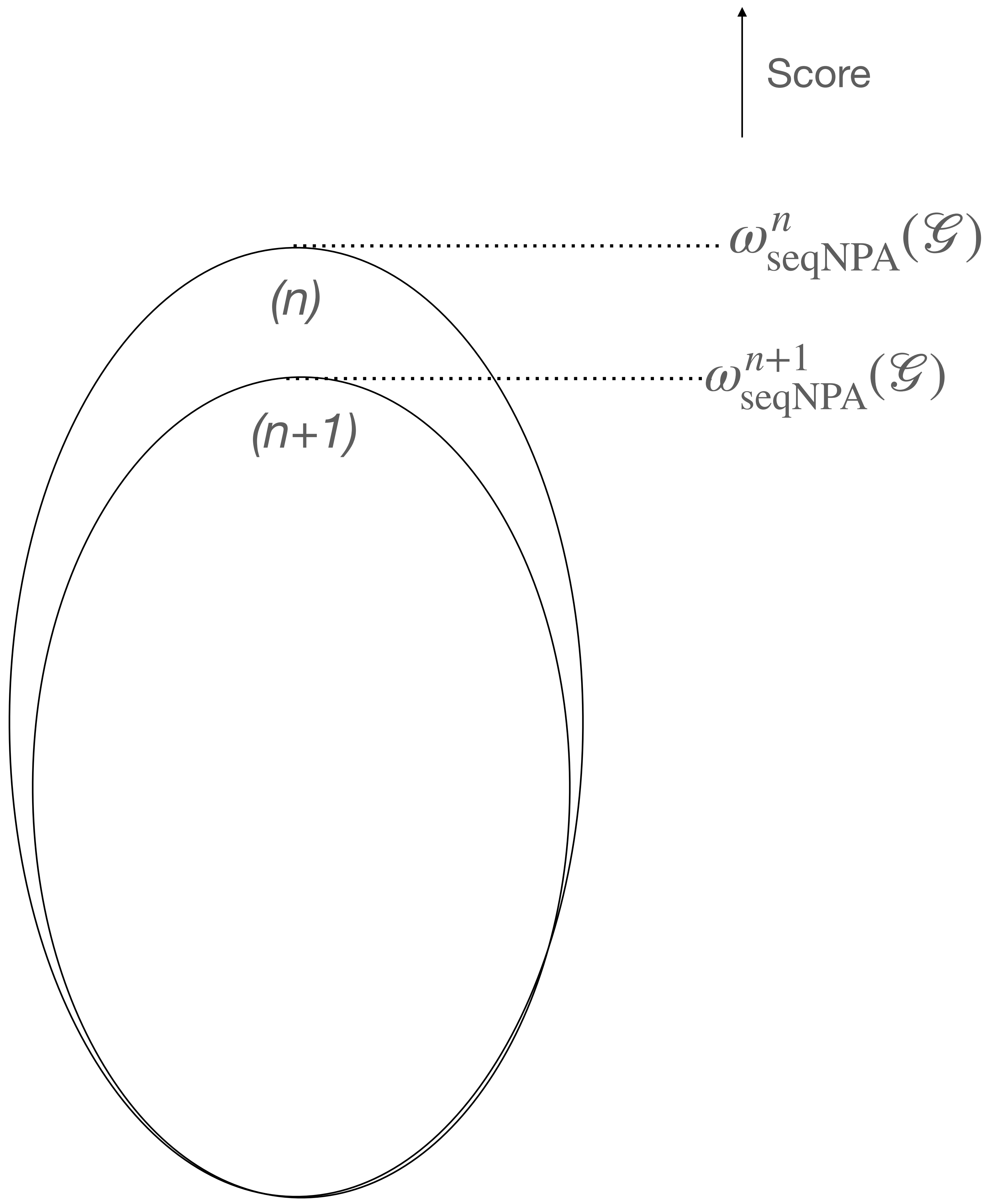

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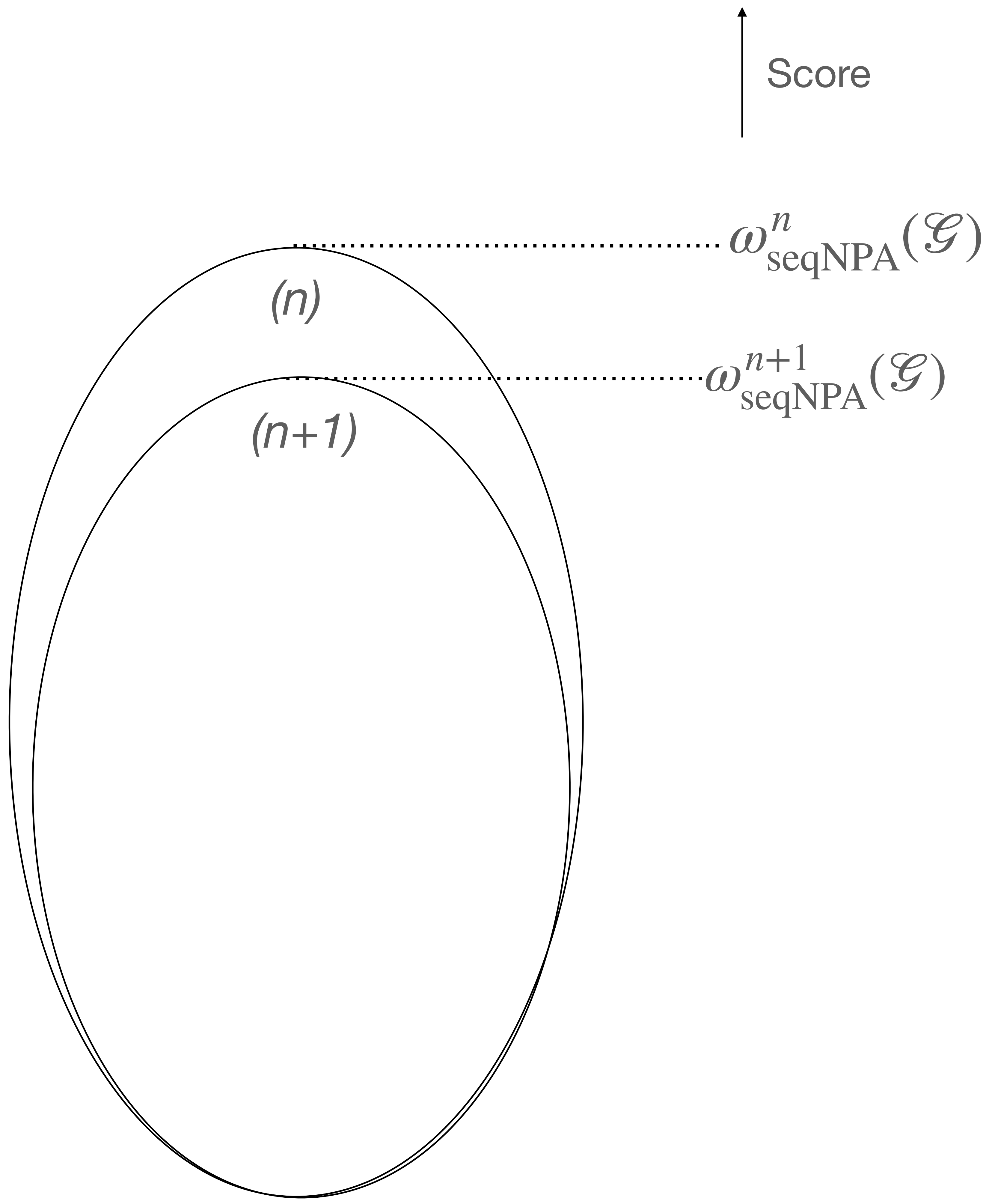

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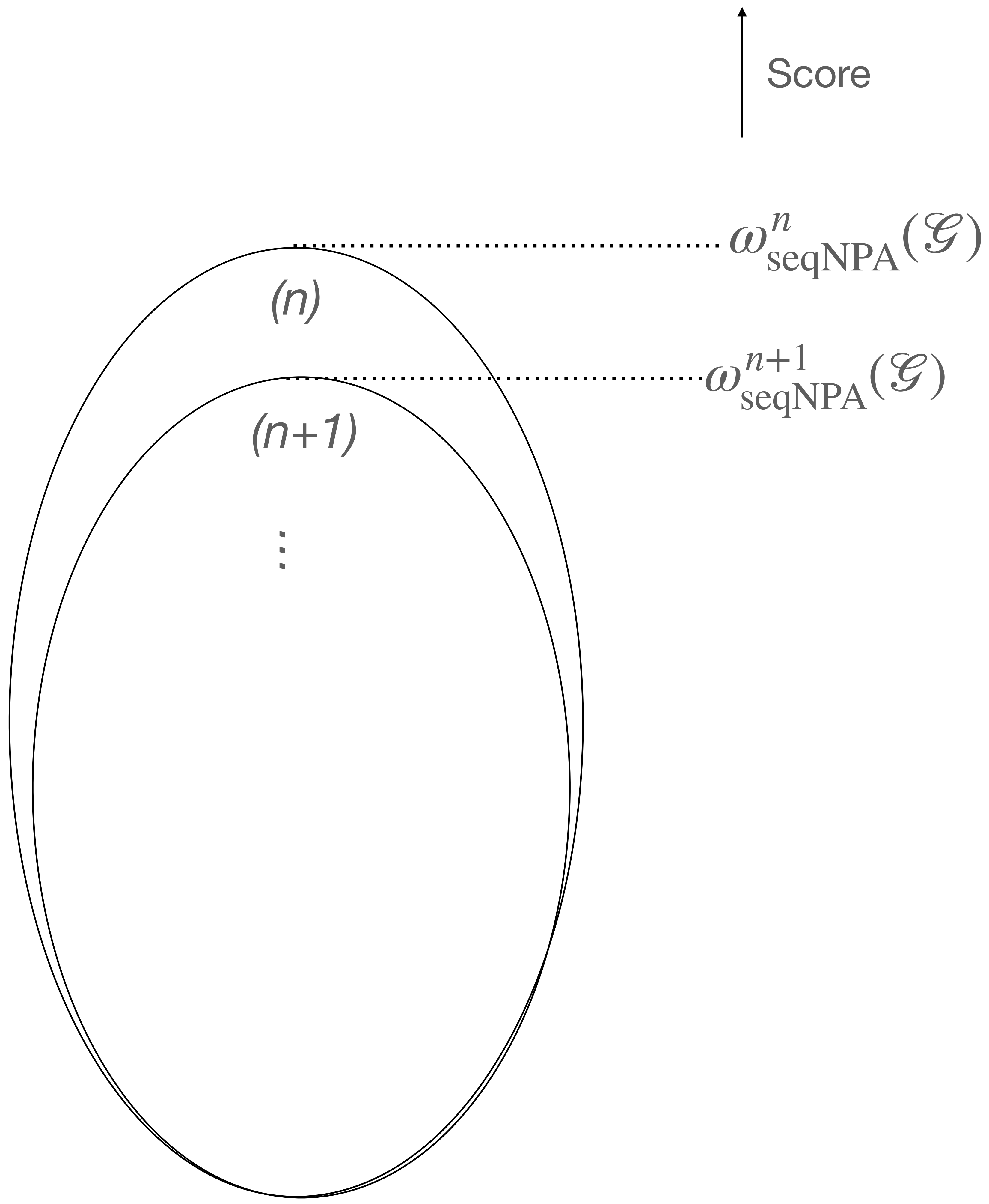

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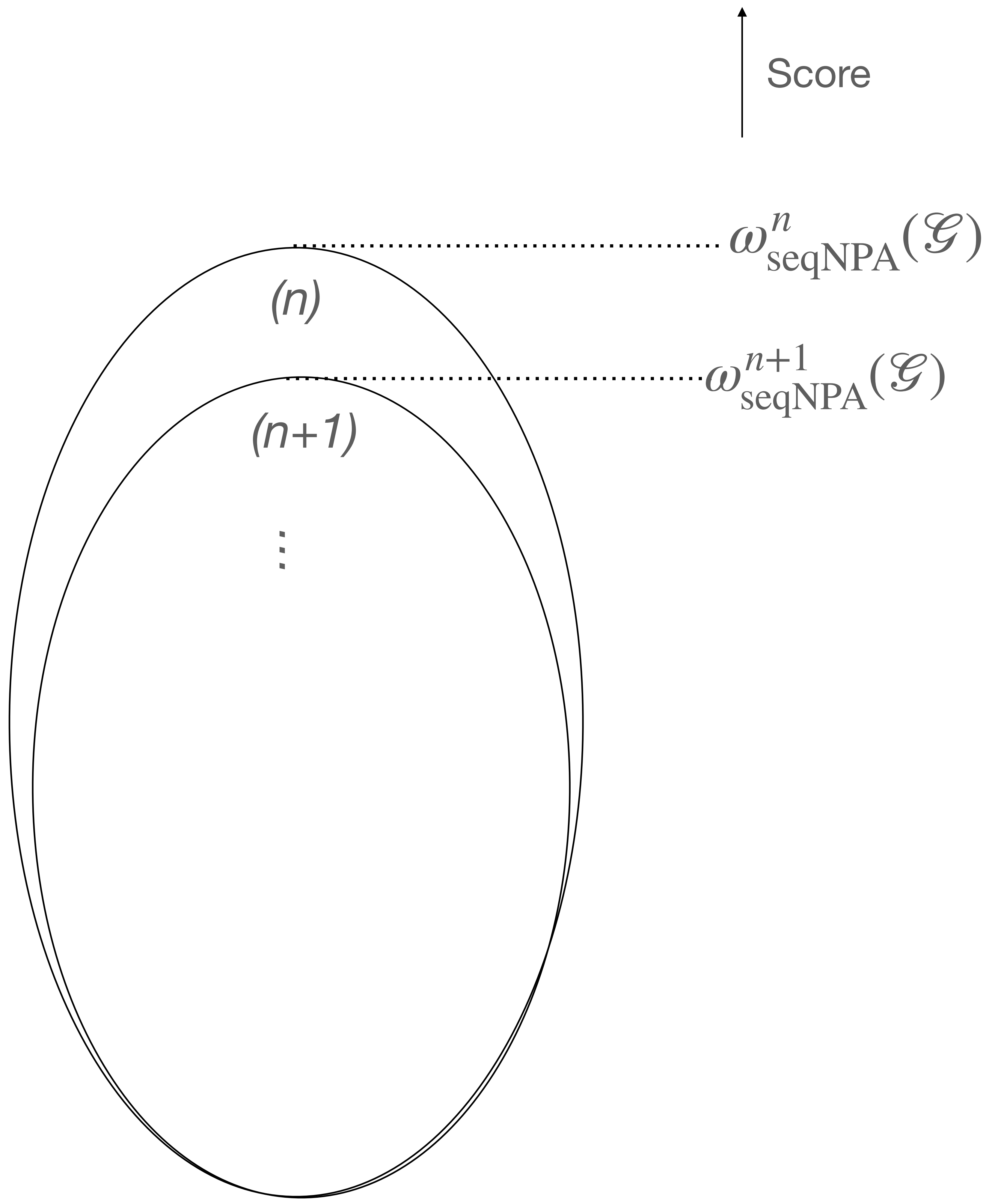

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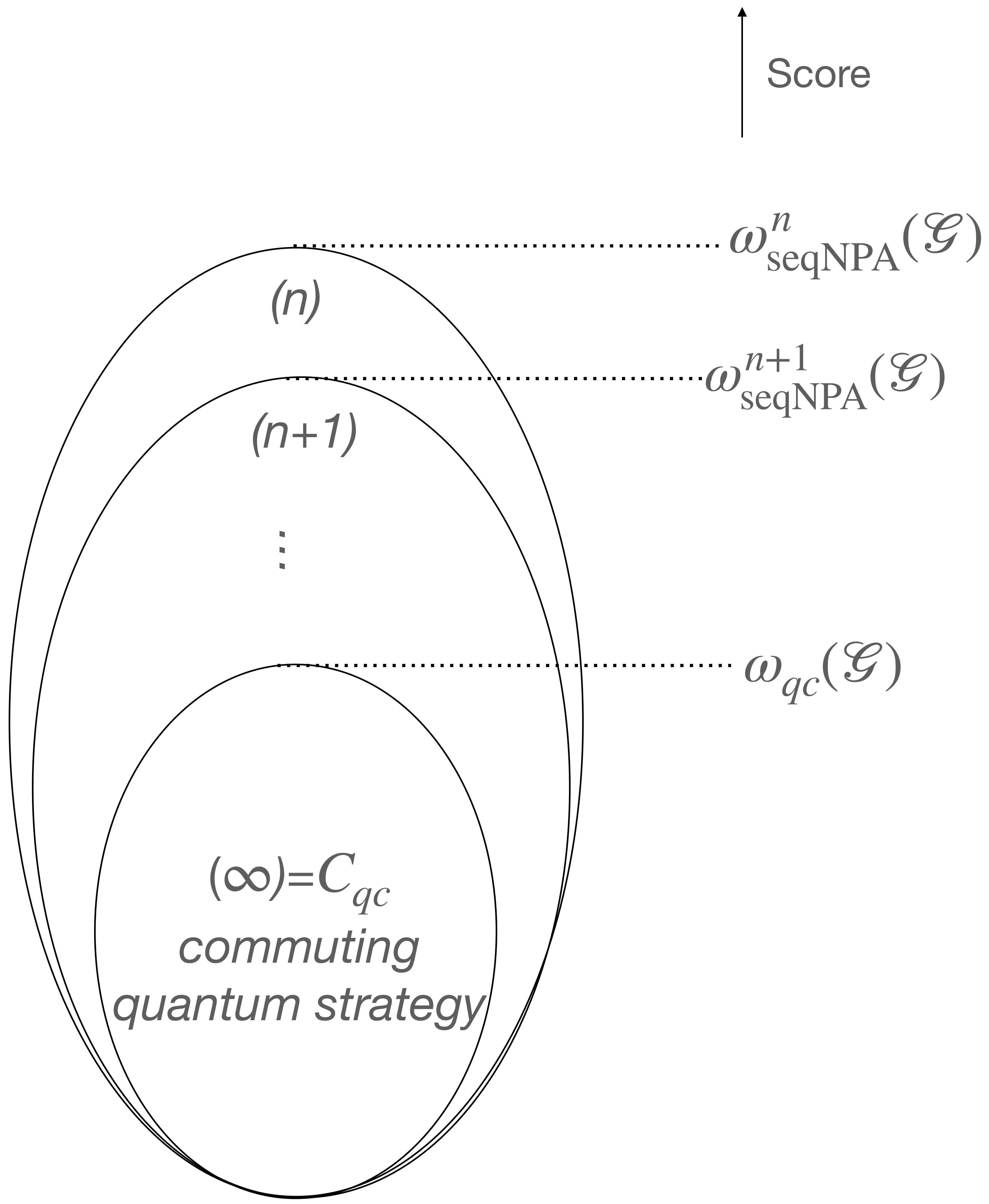

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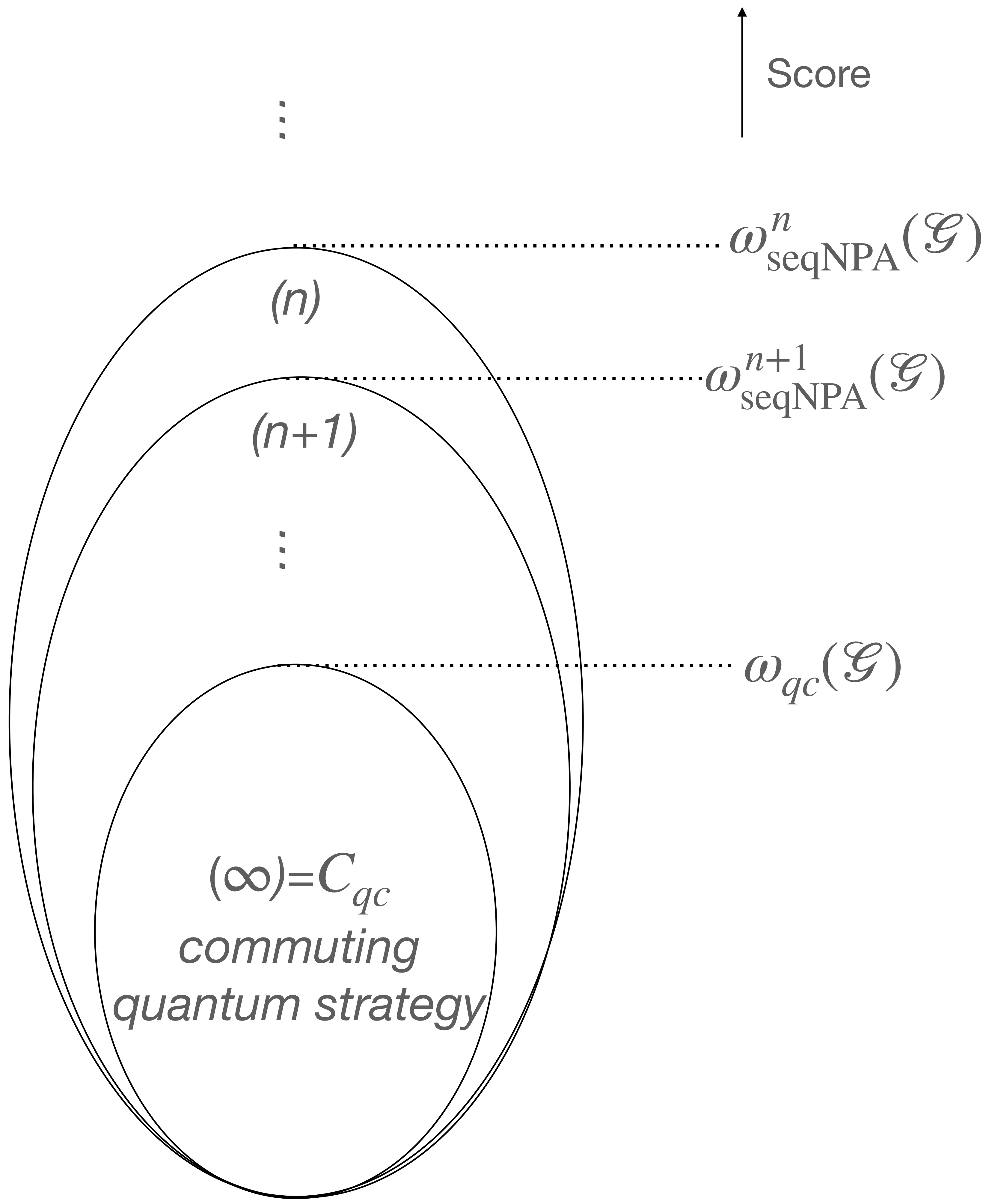

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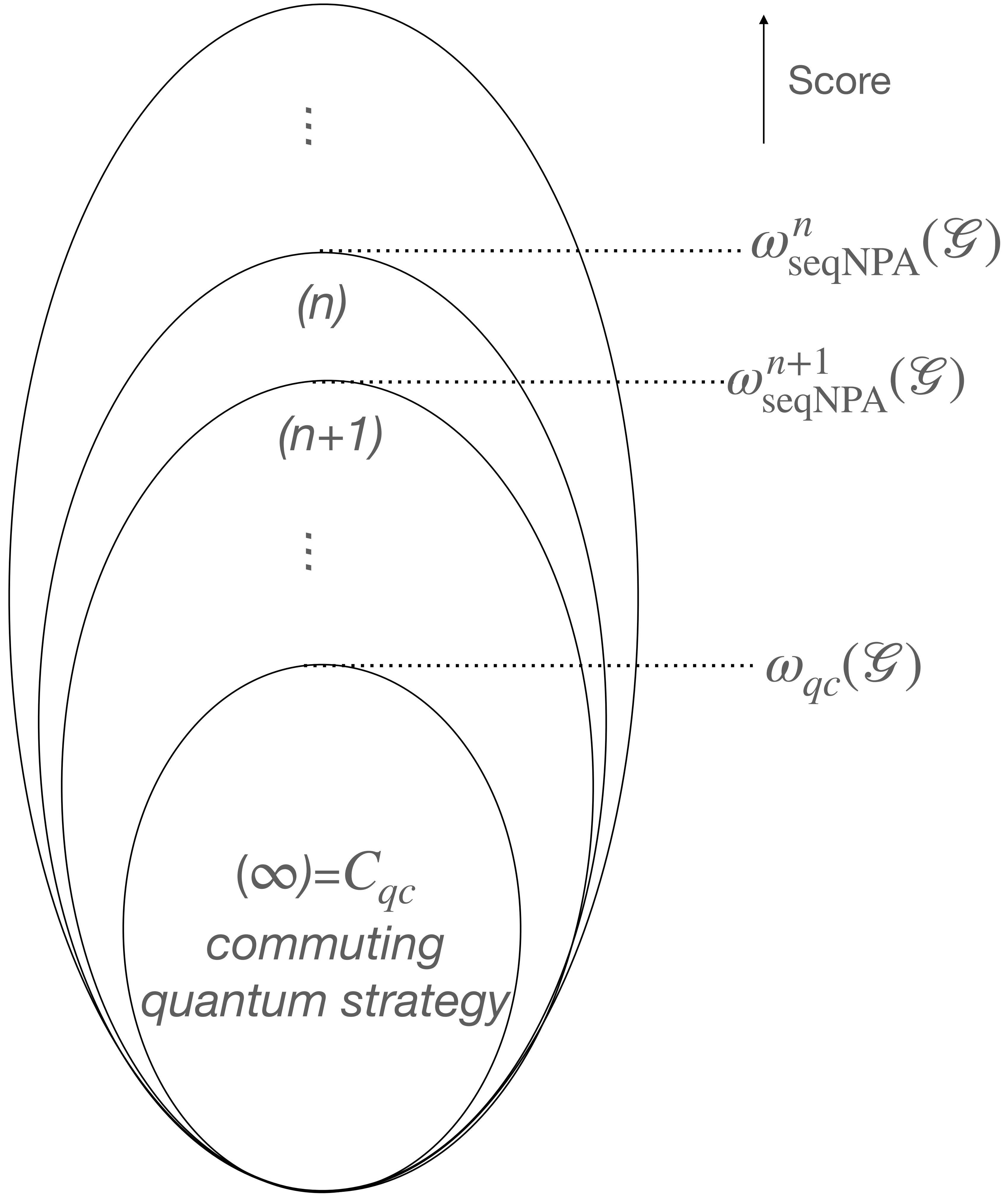

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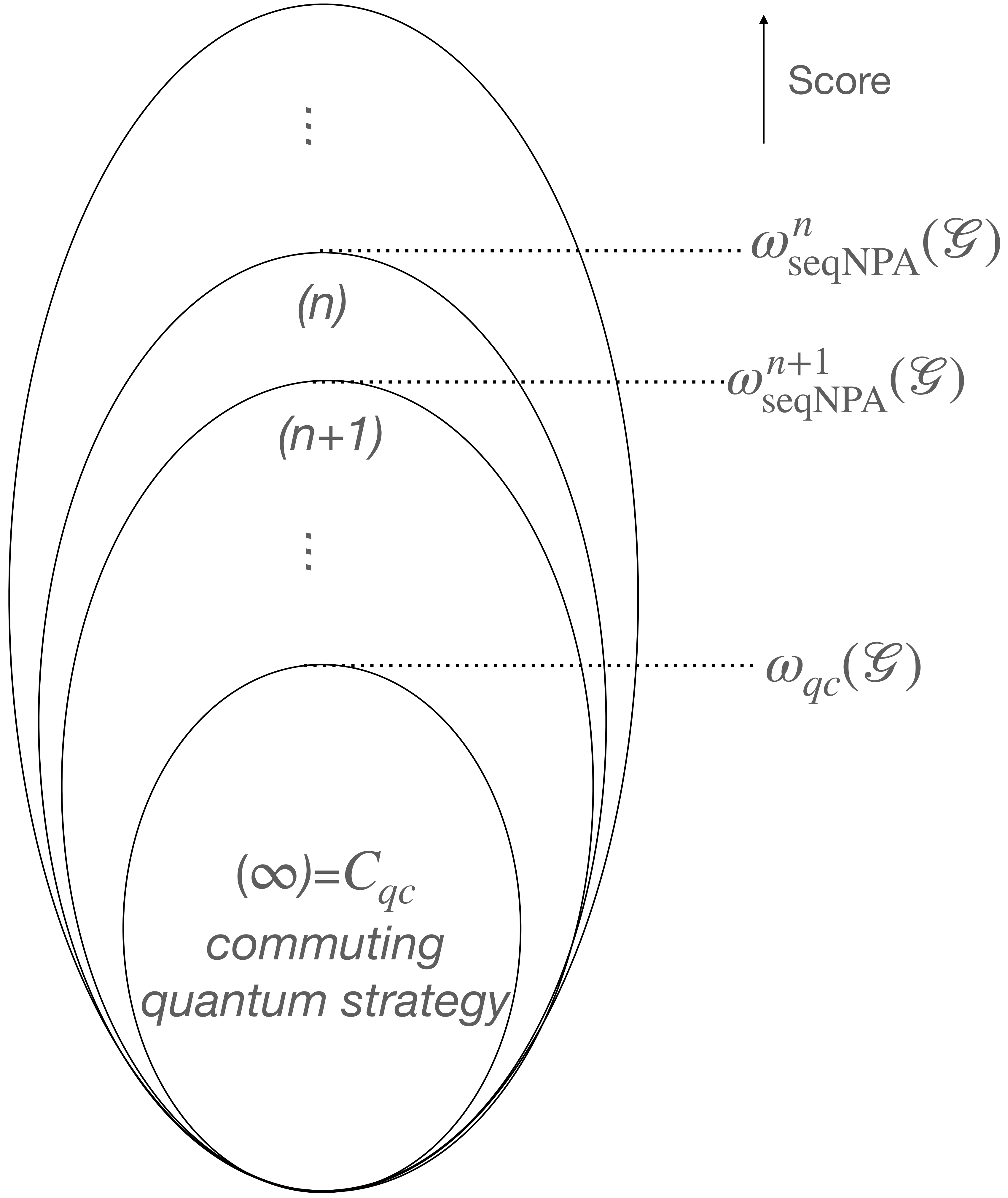

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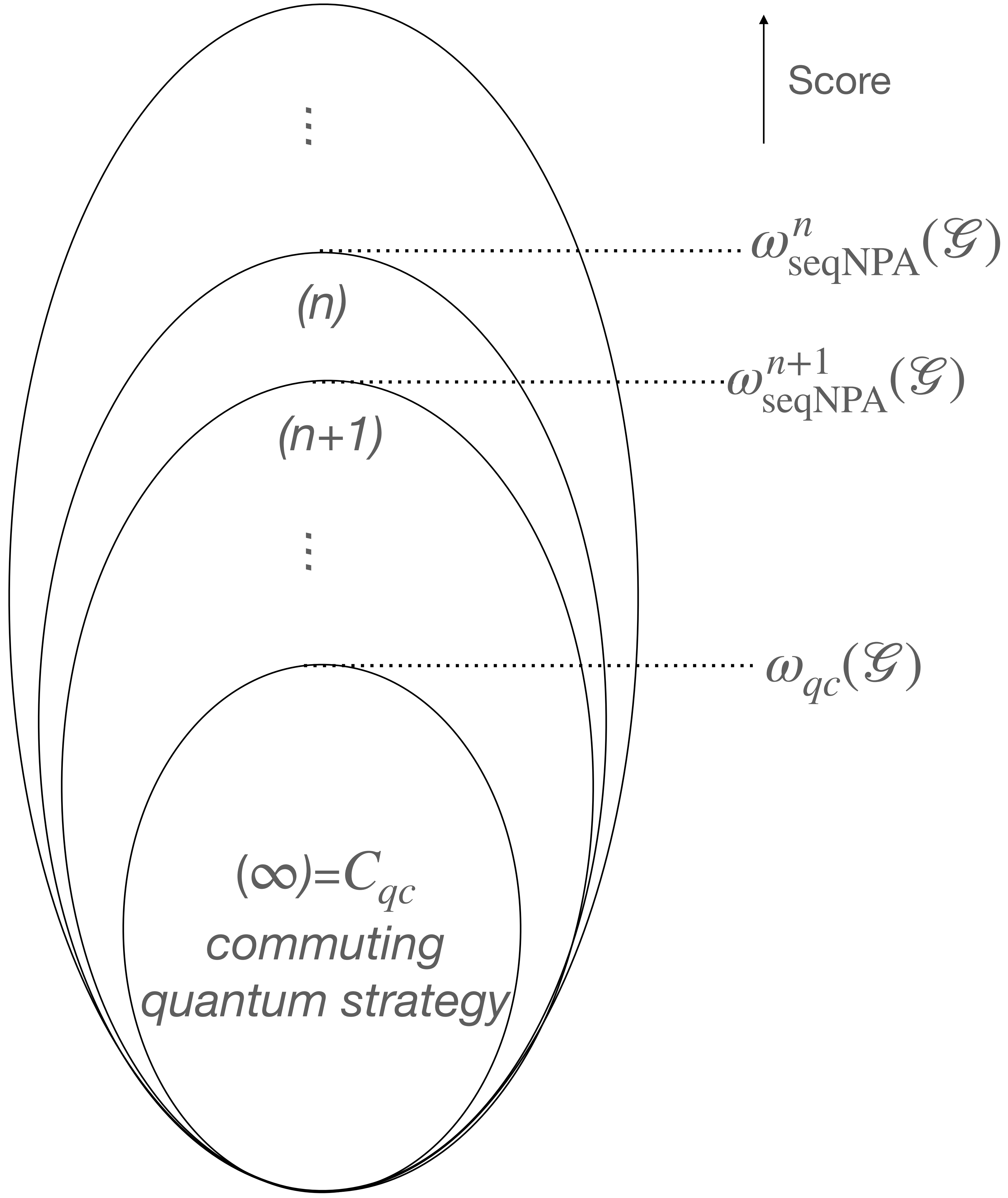

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Solvable by computers!

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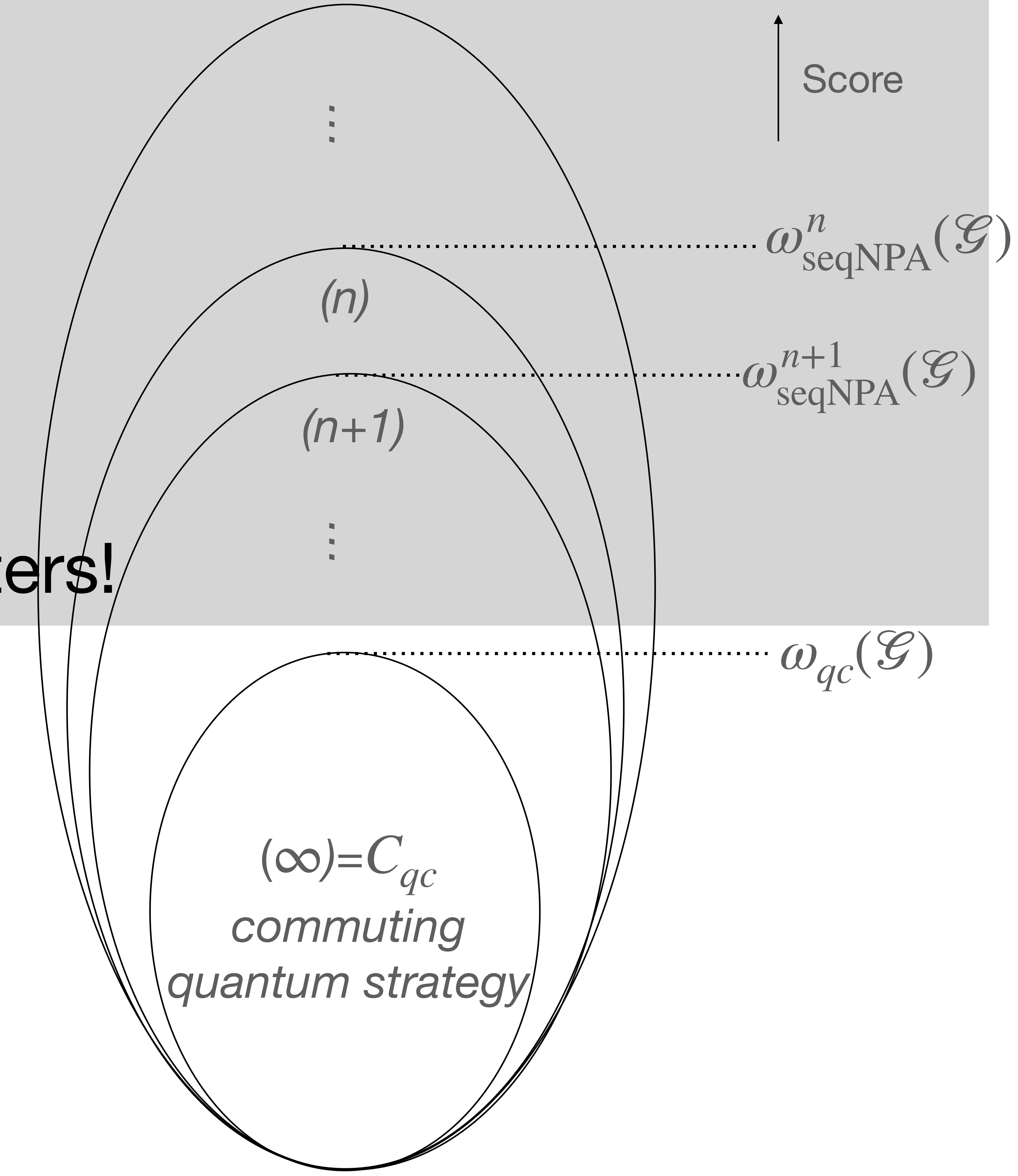

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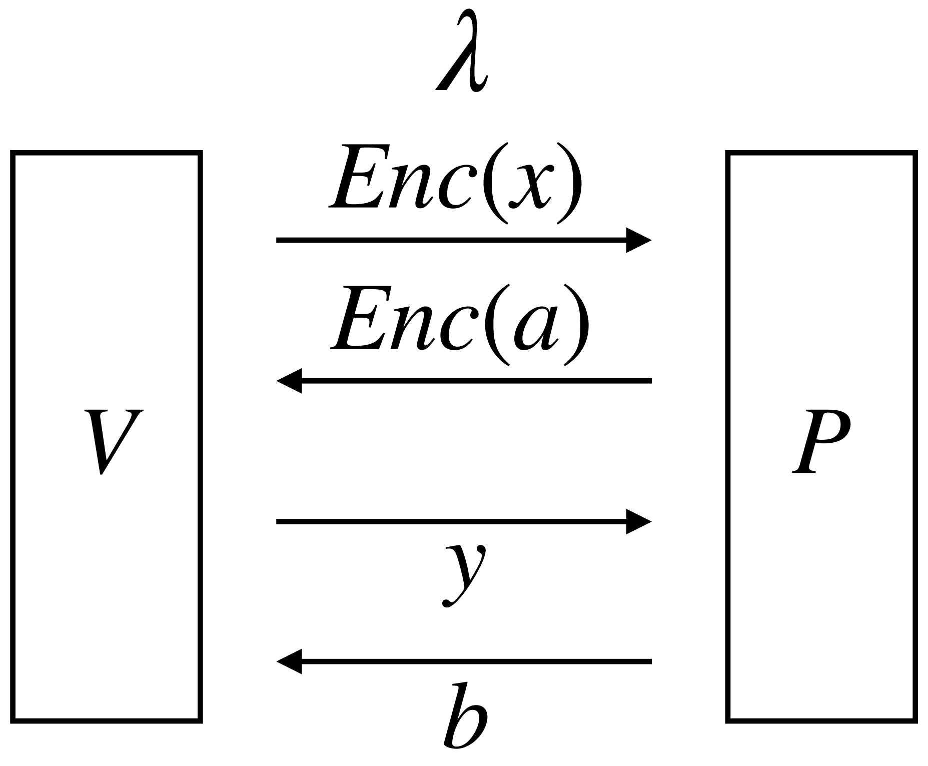
- (Smaller SDP size only in $B_{b|y}$ and no $A_{a|x}$ at the cost of higher NPA level)

Isolate weak signaling



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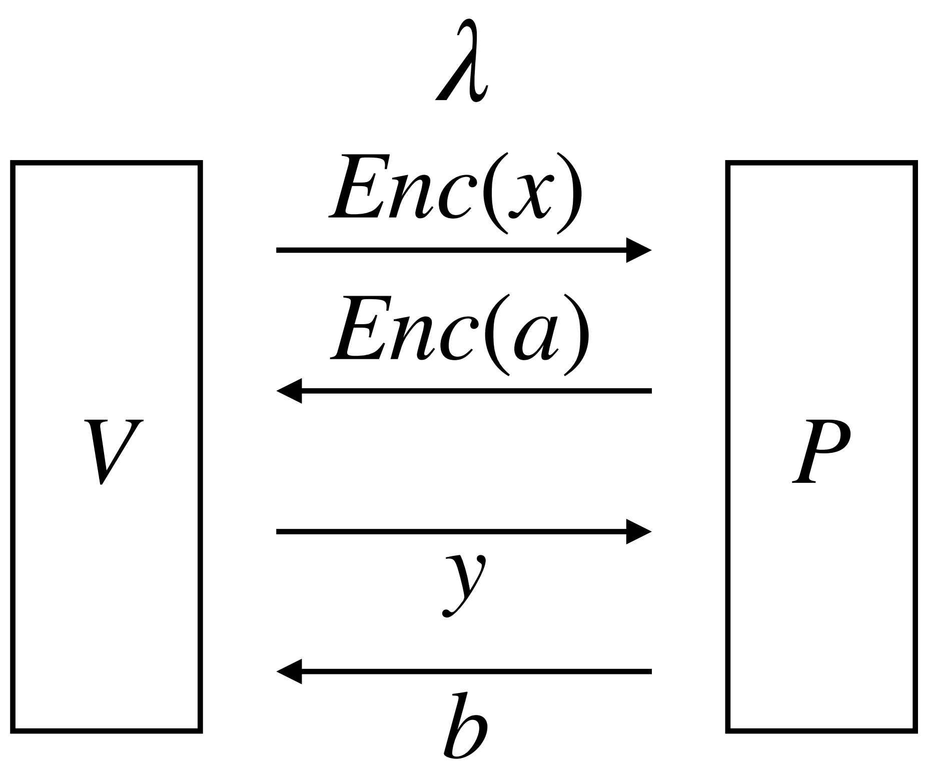

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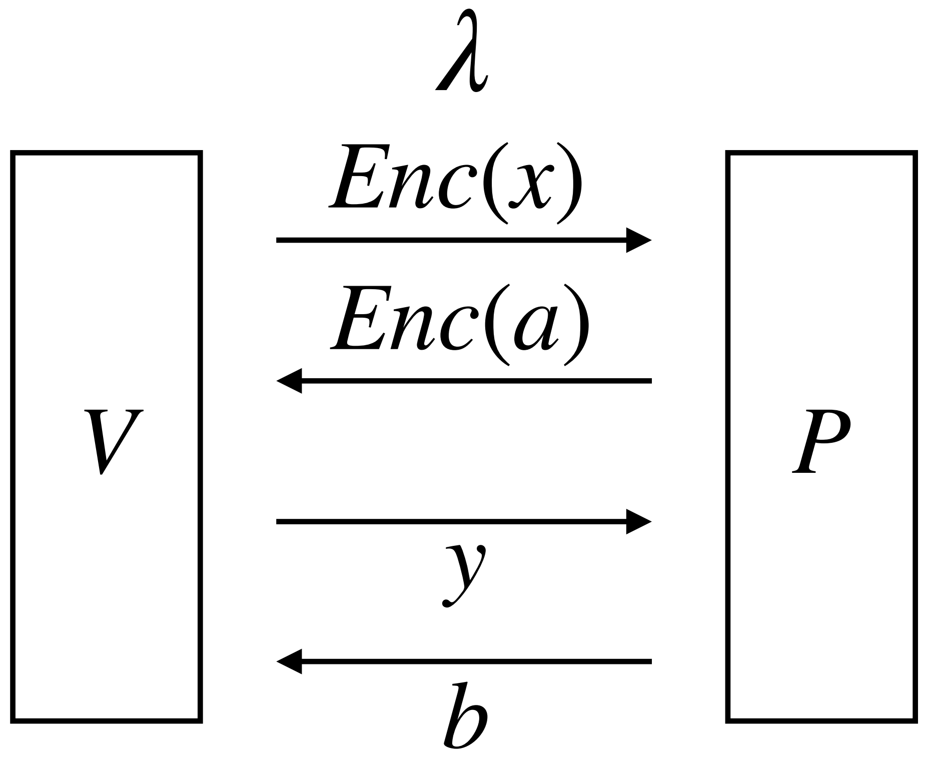

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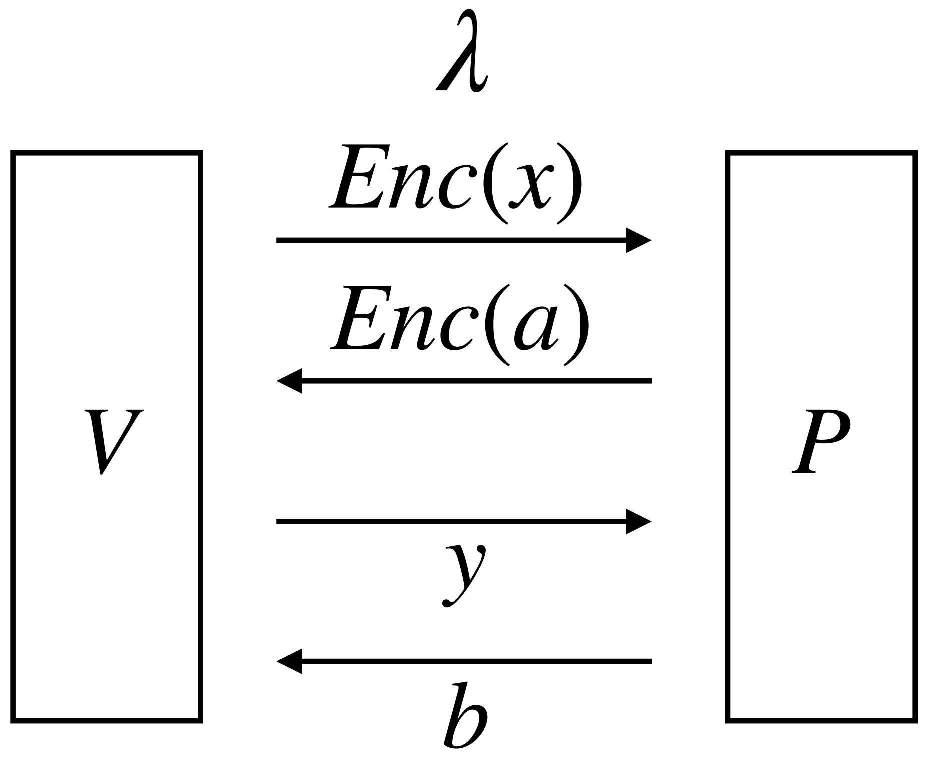

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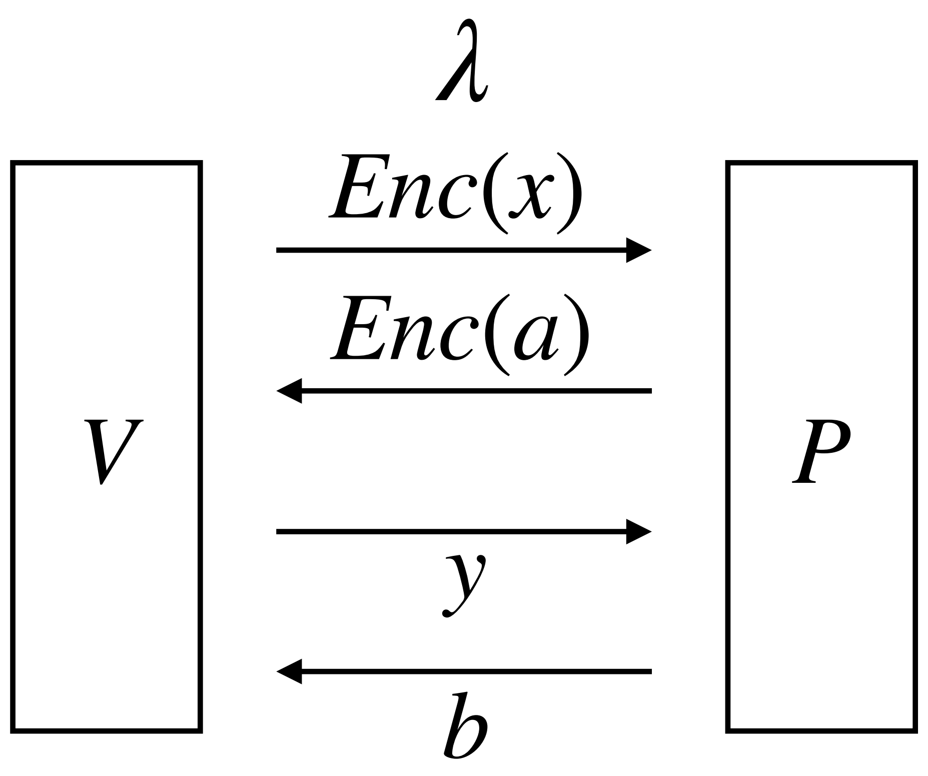

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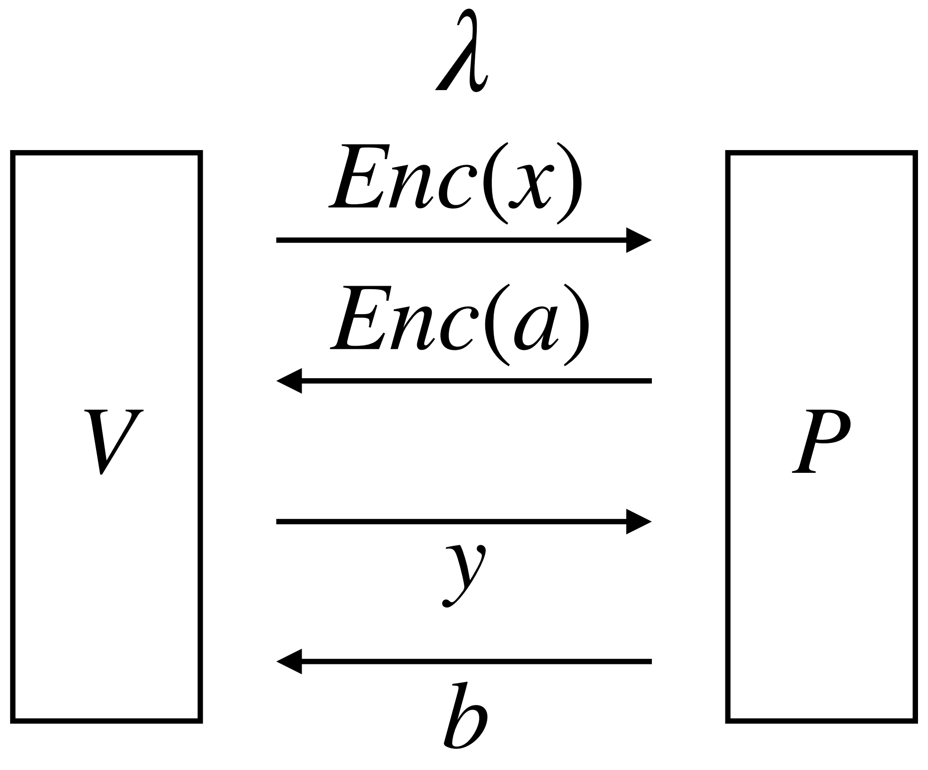
Compiled score

Isolate weak signaling



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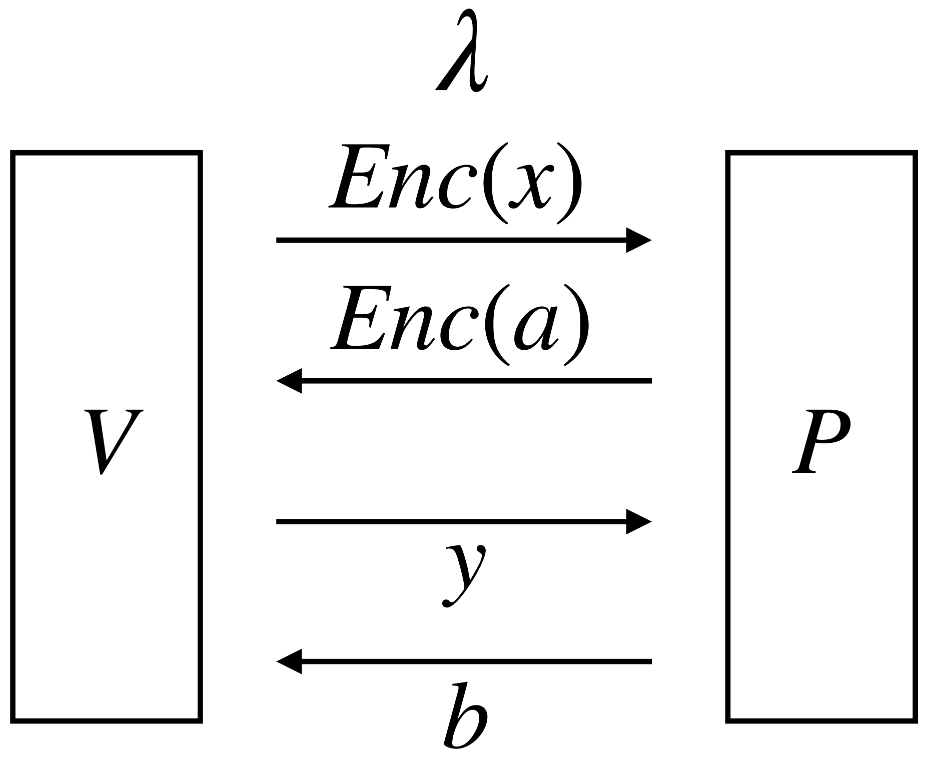

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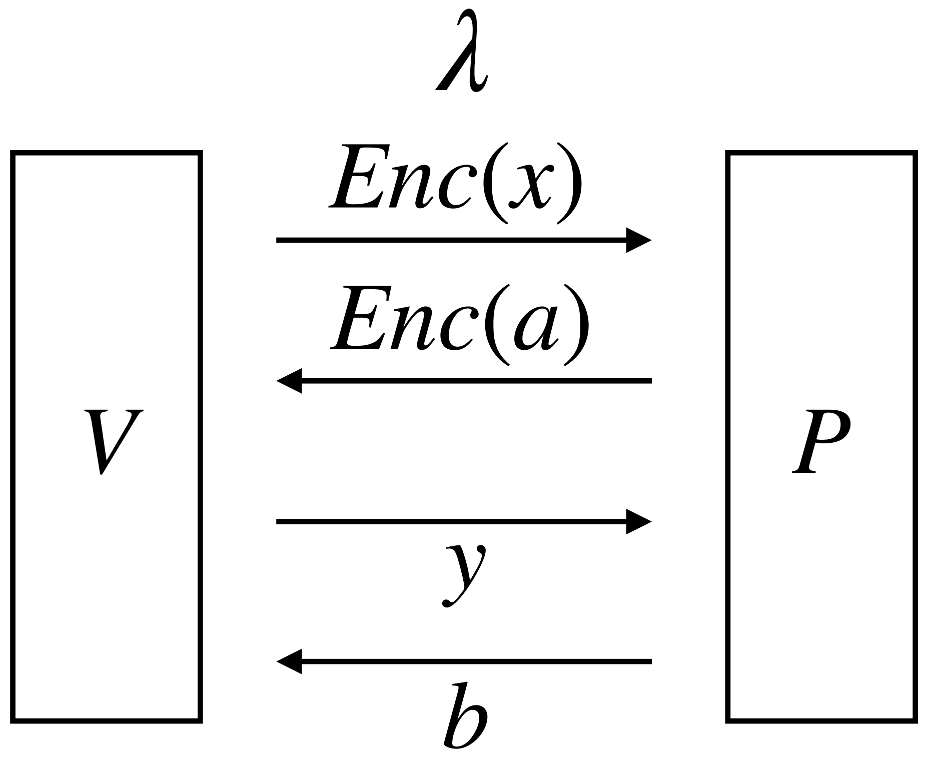
Compiled score

seqNPA!

Isolate weak signaling



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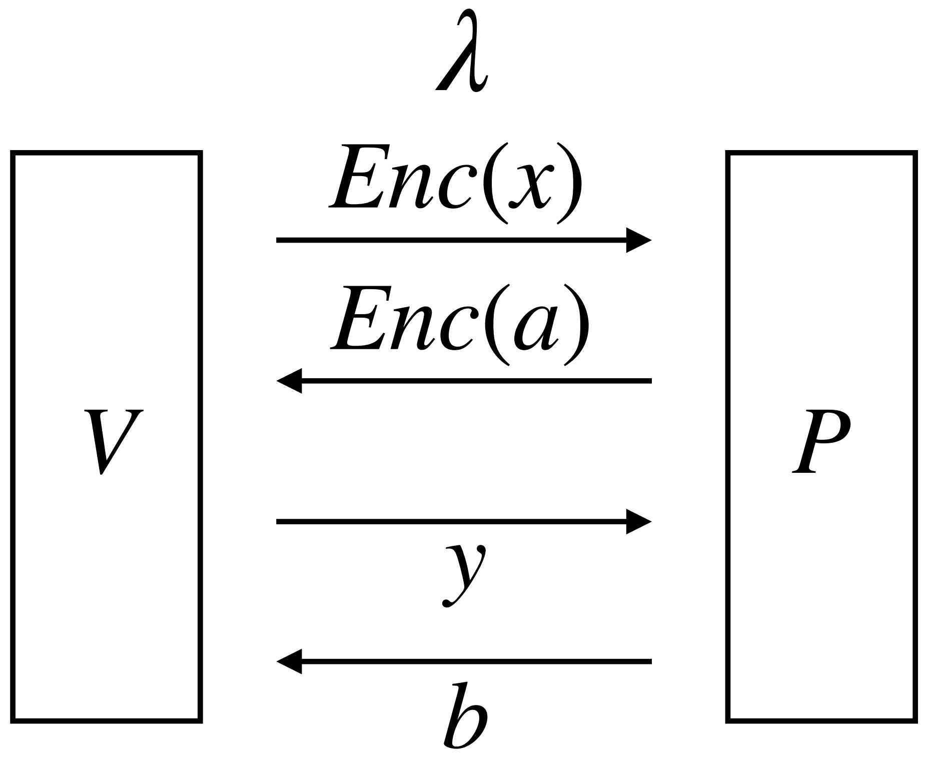

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Main result revisited

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If $\mathcal{G}$ has a finite-dim optimal strategy, then **Main Thm 2** (flatness condition) shows:

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Maybe the dishonest prover can choose to play $\mathscr{G}$ and use S to cheat at $\mathscr{G}_{\mathrm{comp}}$?

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- Need “QHE correctness with auxiliary input for *weakly commuting registers*.”

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  (2) Single provers with cryptographic assumptions (compiled games)
- Potential to translate other space-like separated protocols?

Reference

- [Our results] *"Quantitative quantum soundness for bipartite compiled Bell games via the sequential NPA hierarchy"* Igor Klep, Connor Paddock, Marc-Olivier Renou, Simon Schmidt, Lucas Tendick, Xiangling Xu, Yuming Zhao

- [KLVY23] *Quantum advantage from any non-local game.* Y. Kalai, A. Lombardi, V. Vaikuntanathan, L. Yang

- [KMPSW24] *A bound on the quantum value of all compiled nonlocal games*. A. Kulpe, G. Malavolta, C. Paddock, S. Schmidt, M. Walter

- [KMP22] *Sparse noncommutative polynomial optimization.* Igor Klep, Victor Magron, and Janez Povh

Bounding the asymptotic quantum value of all multipartite compiled nonlocal games

Matilde Baroni

with Dominik Leichtle, Siniša Janković, Ivan Šupić

From 2 to 3 : why do we care

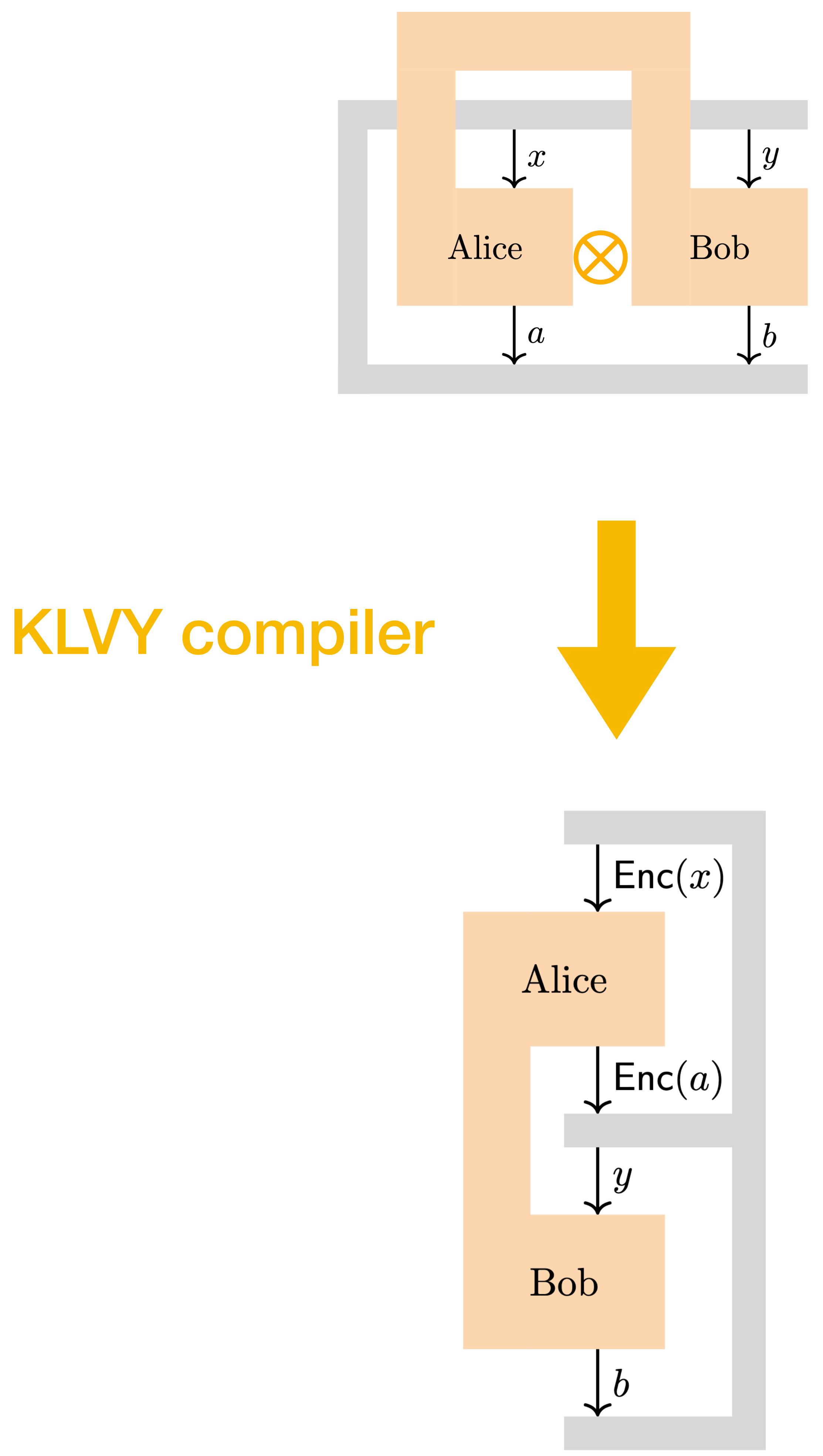
1. For classical it has already been done

2. Multipartite (>2) correlations are interesting (e.g. post-quantum steering)

3. Crypto applications

4. Space-like separation for multiple players is problematic

KMPSW techniques



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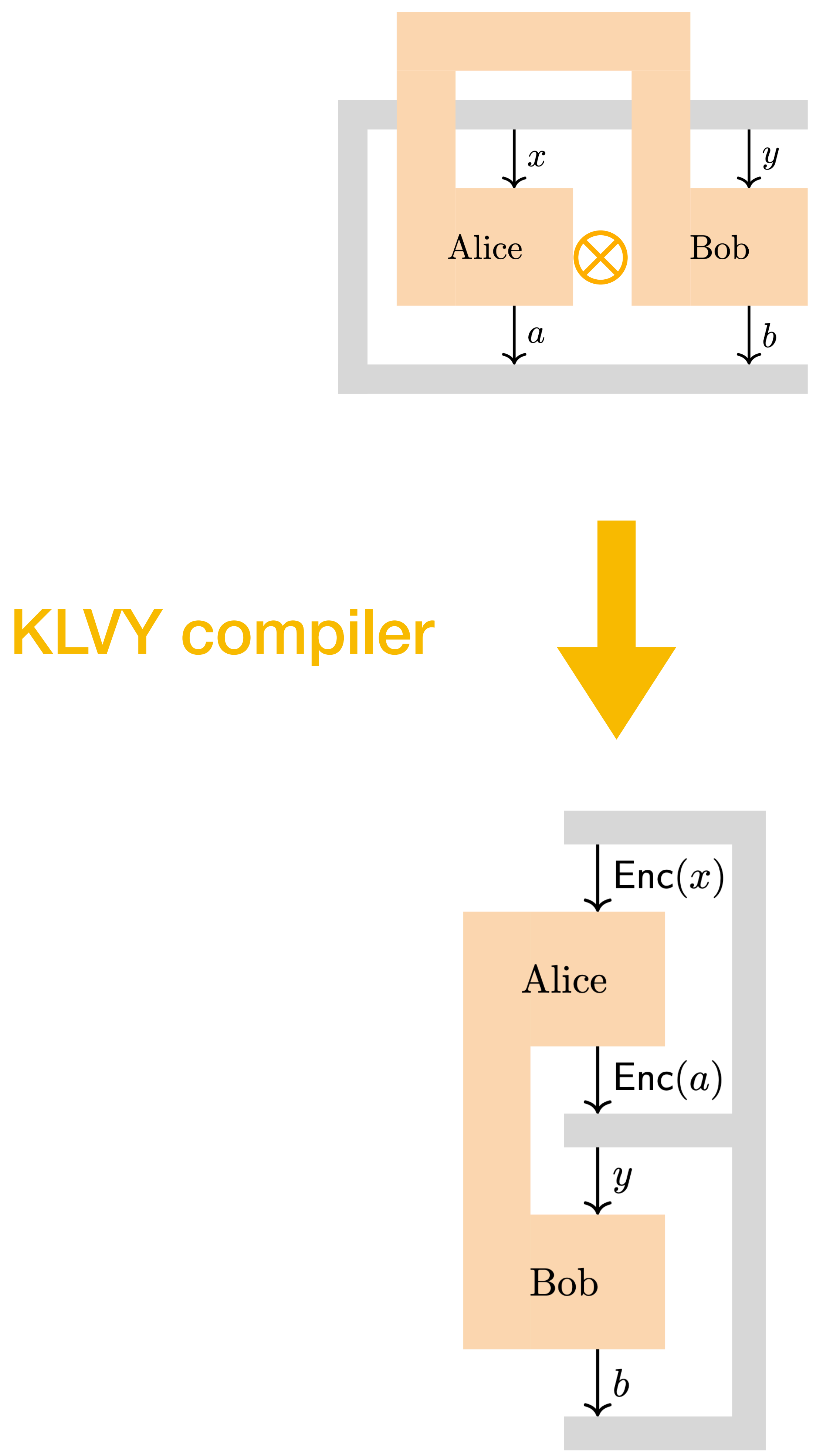
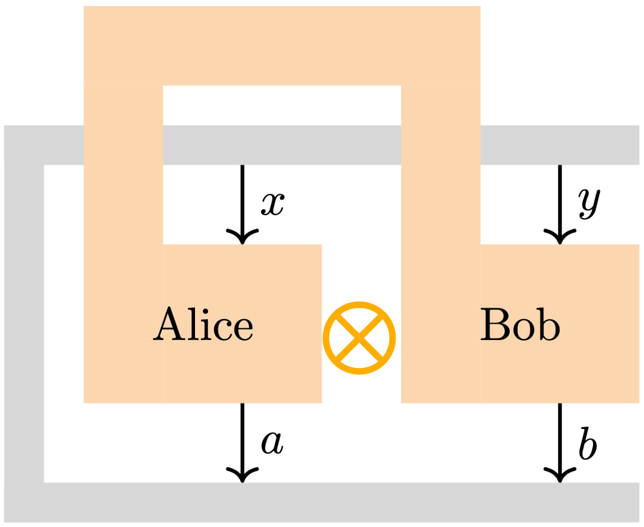
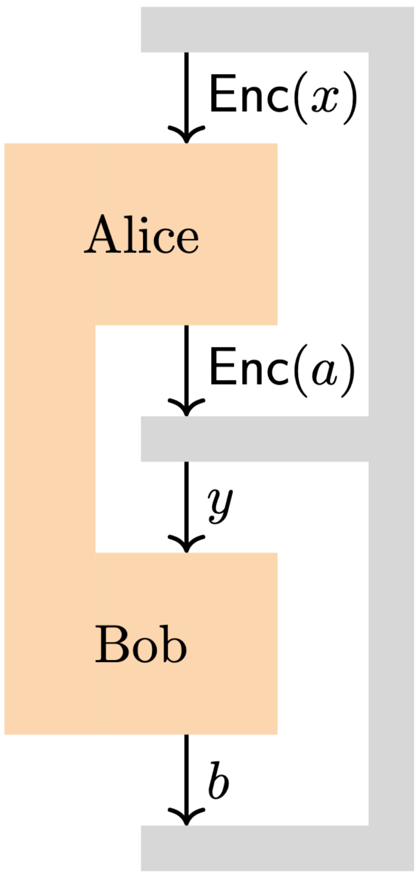


Sequential game

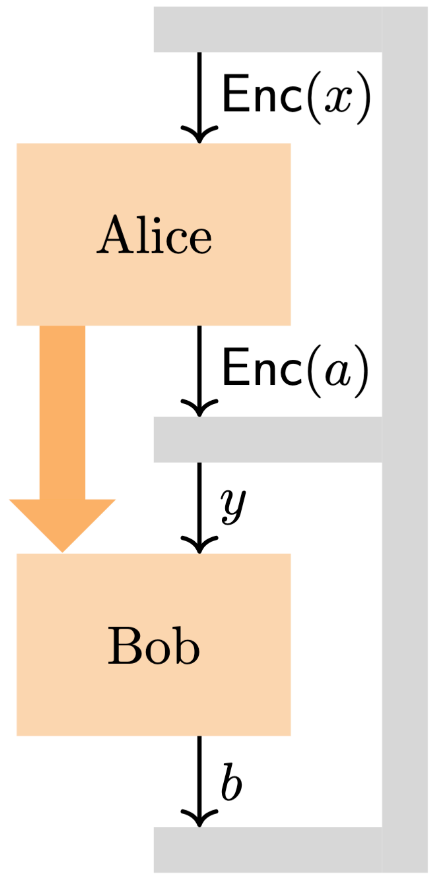
KMPSW techniques



KLVY compiler

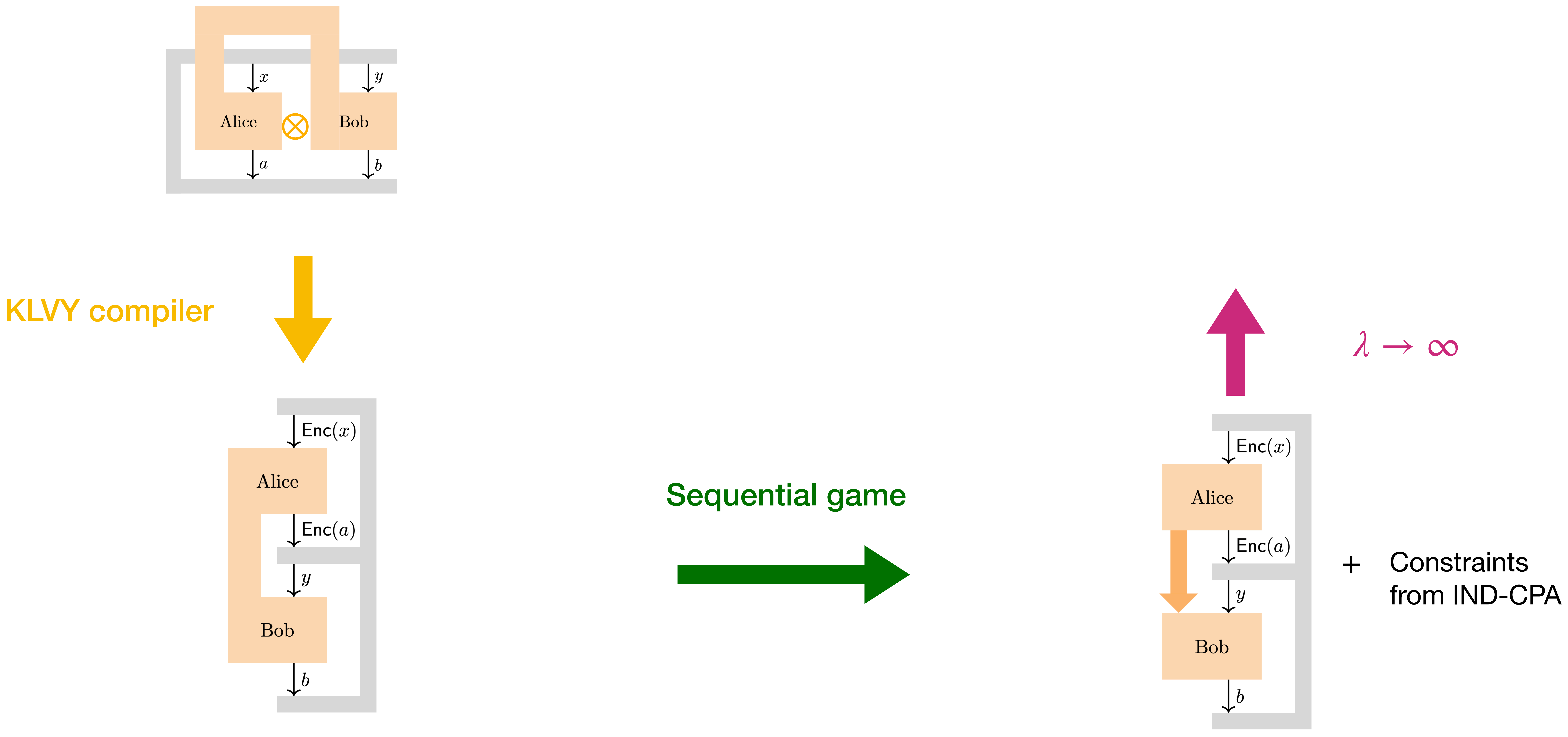


Sequential game



+ Constraints from IND-CPA

KMPSW techniques



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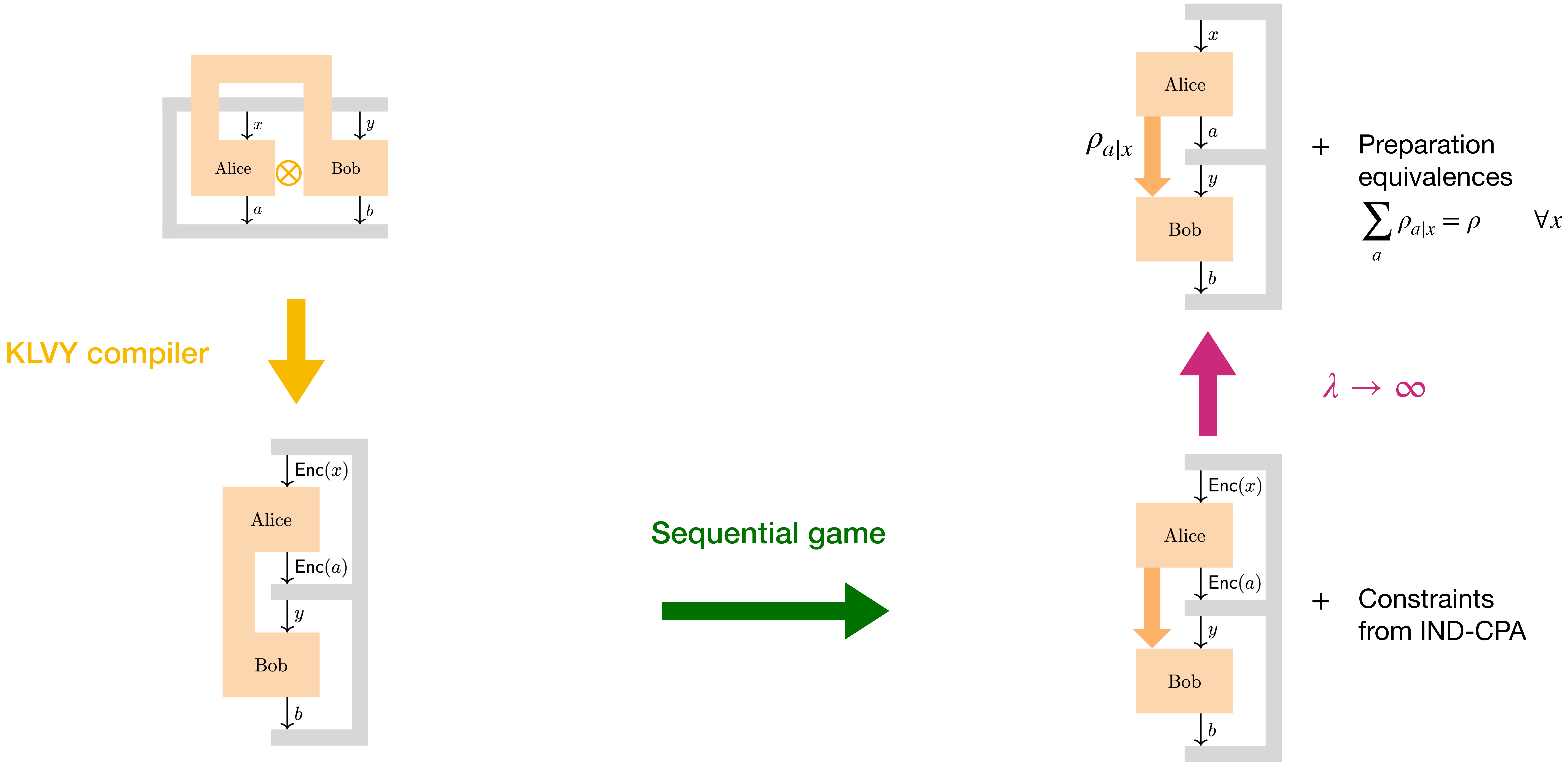

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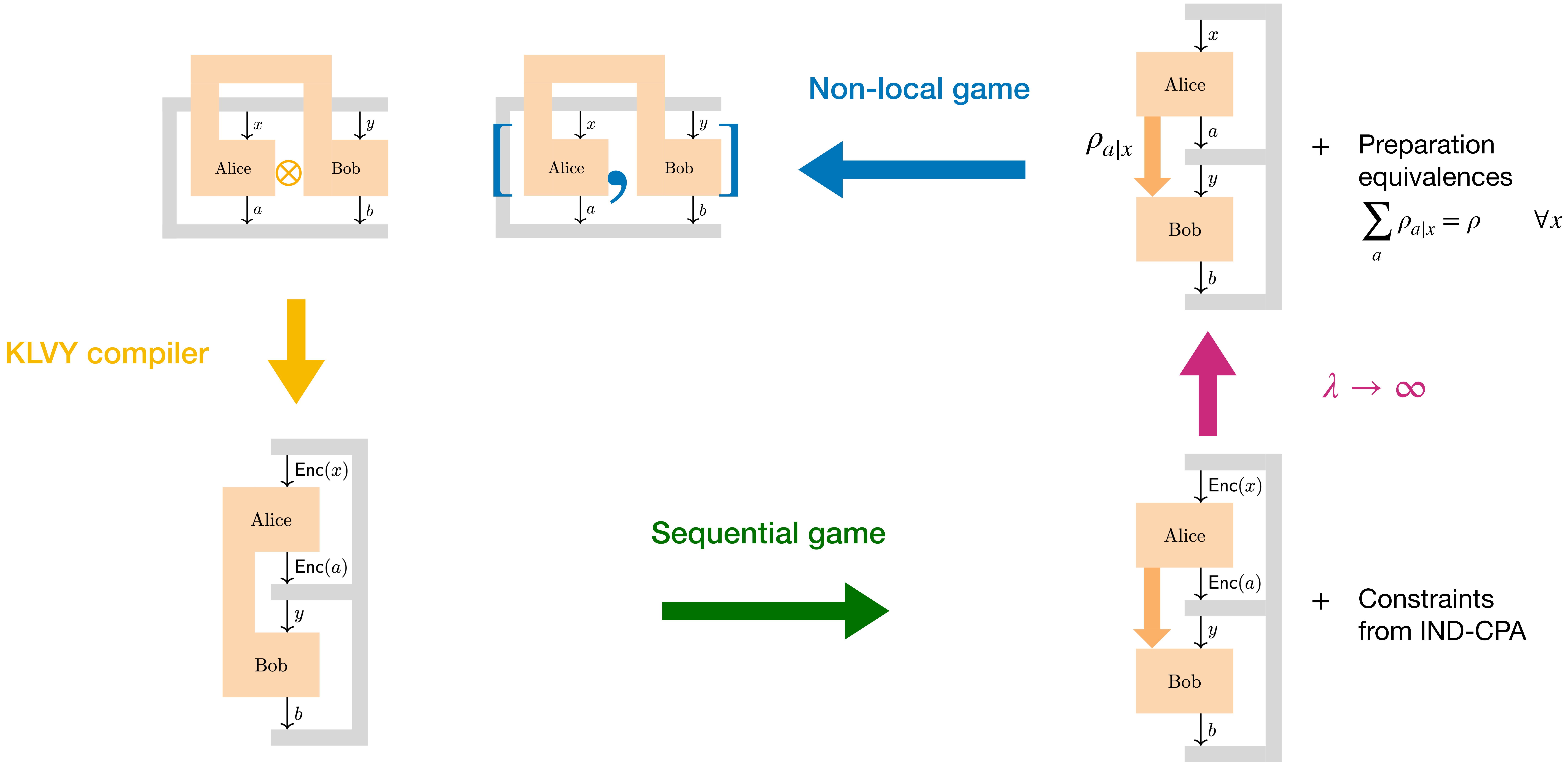
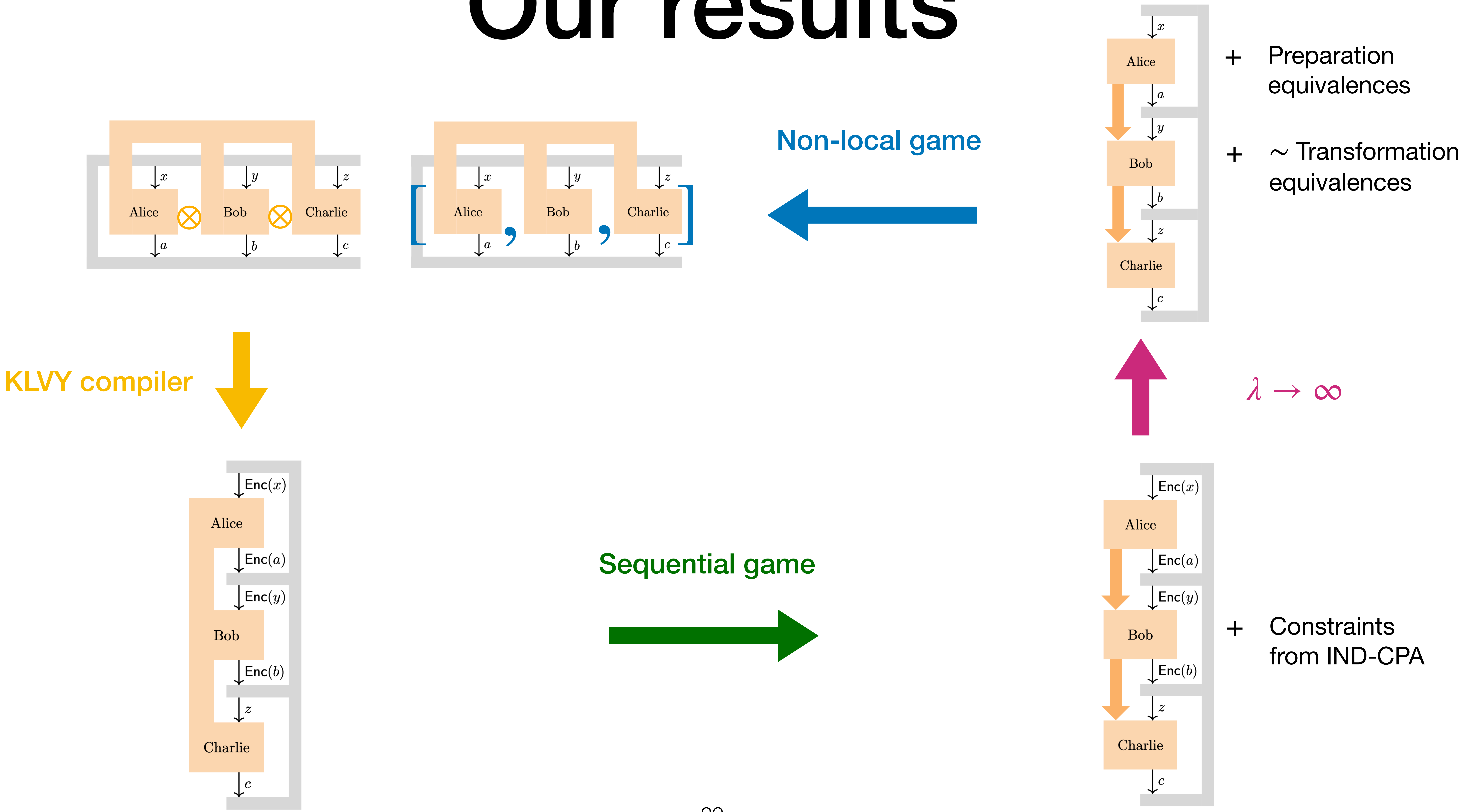


Our results


Alice
Bob
Charlie
Non-local game
KLVY compiler
Enc(x)
Enc(a)
Enc(y)
Enc(b)
Sequential game
+ Preparation equivalences
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λ → ∞
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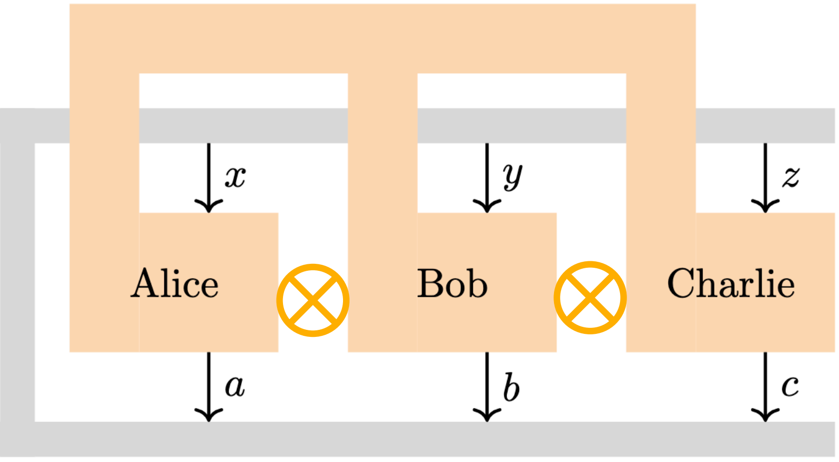
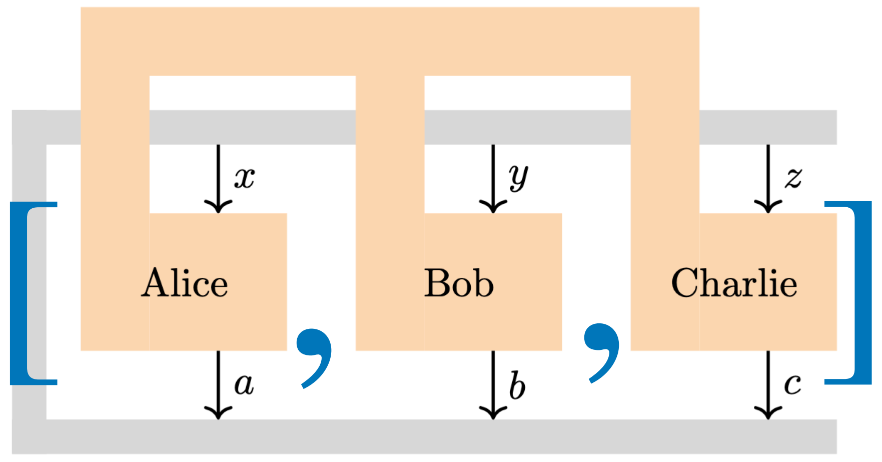
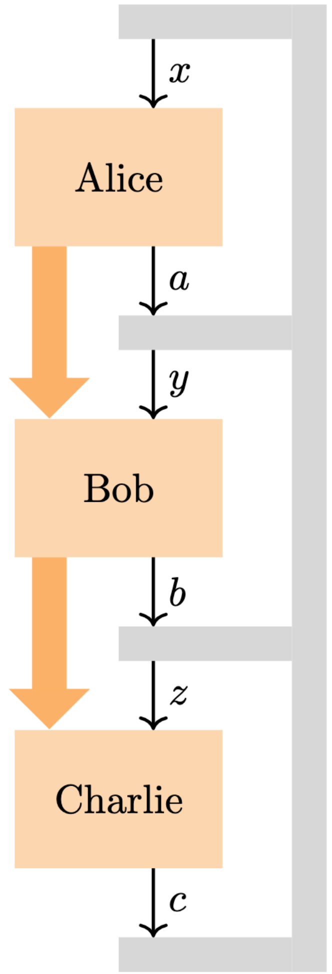




Non-local game



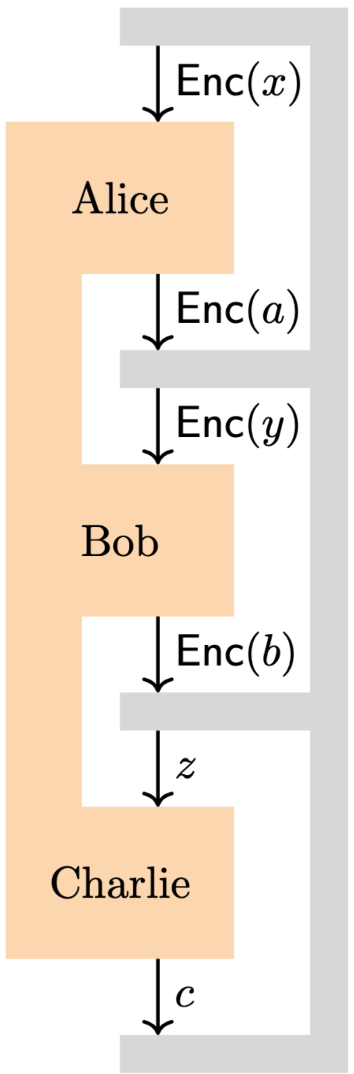
\+ Preparation equivalences

\+ $\sim$ Transformation equivalences

KLVY compiler

For all k-players games !

$\lambda \to \infty$



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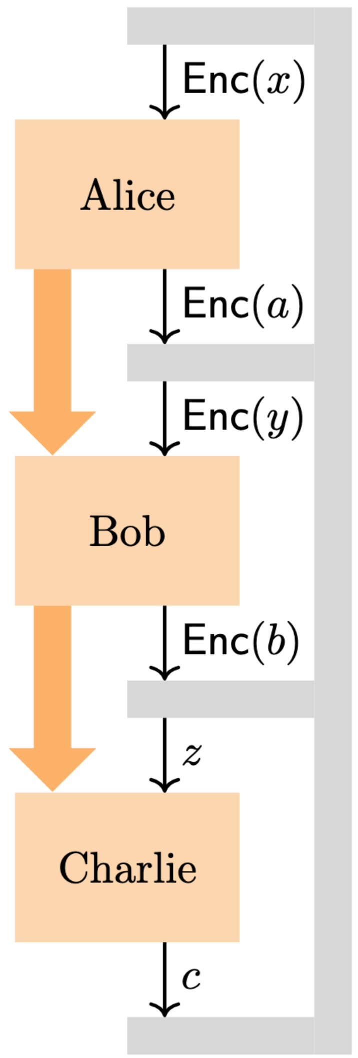

\+ Constraints from IND-CPA

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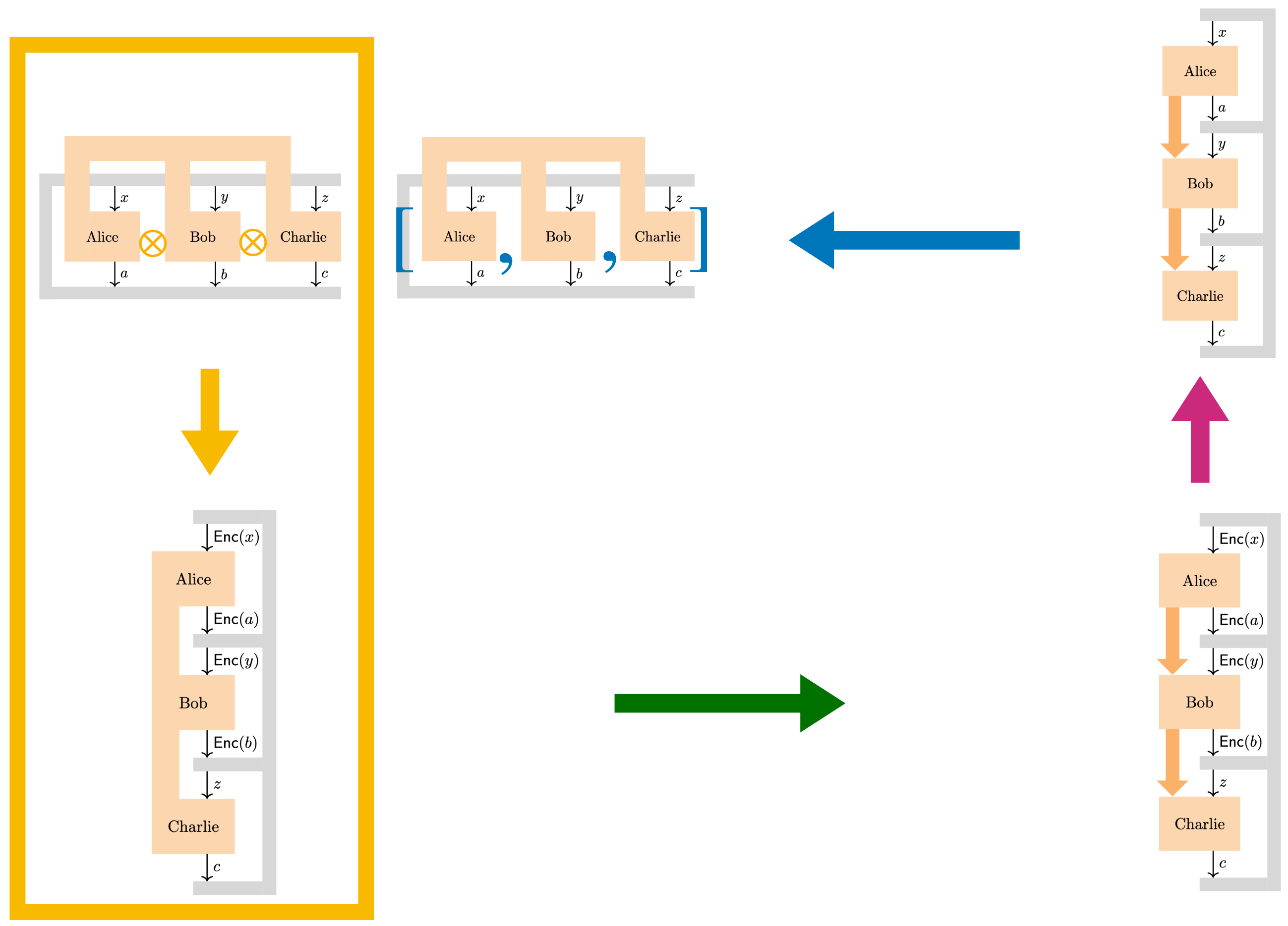
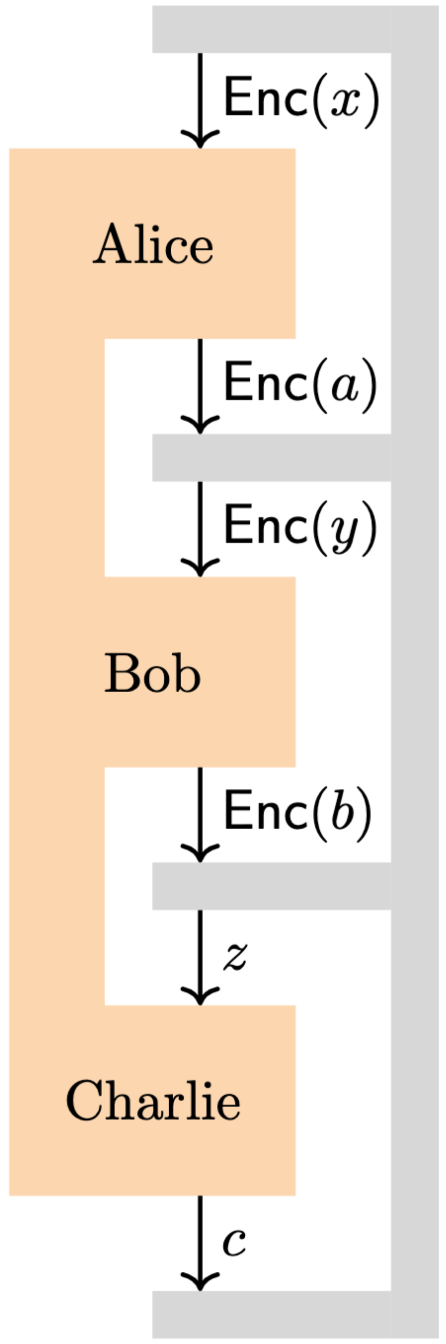


The first two interactions are encrypted, the third is in the clear

Classical soundness

Quantum completeness

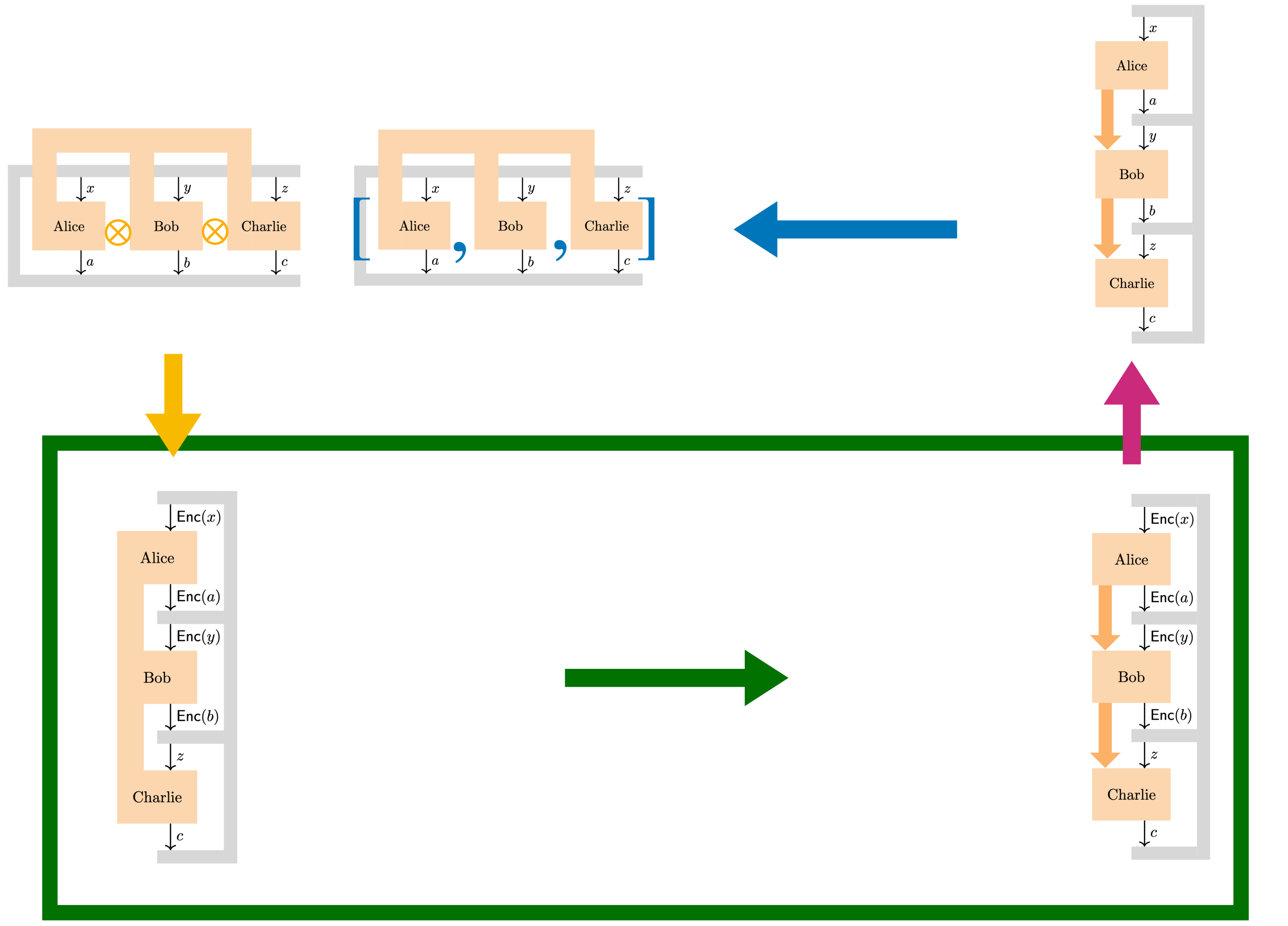
Quantum soundness ?

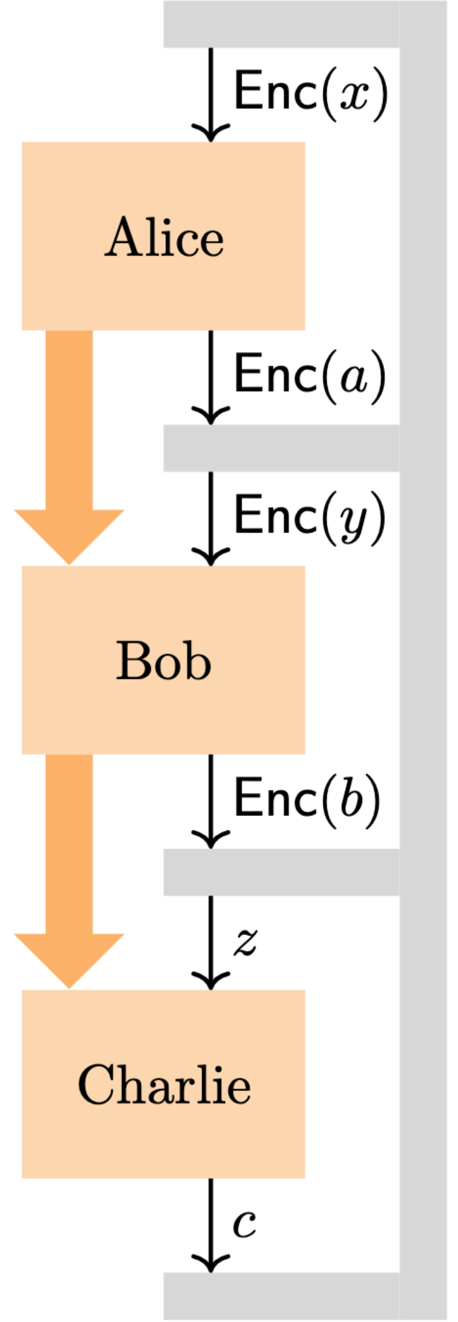


2. Constraints on the correlations

Quantum strategies

$$p_\lambda(a,b,c|x,y,z) = \mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{b|y}(\rho^\lambda_{a|x})\right]$$





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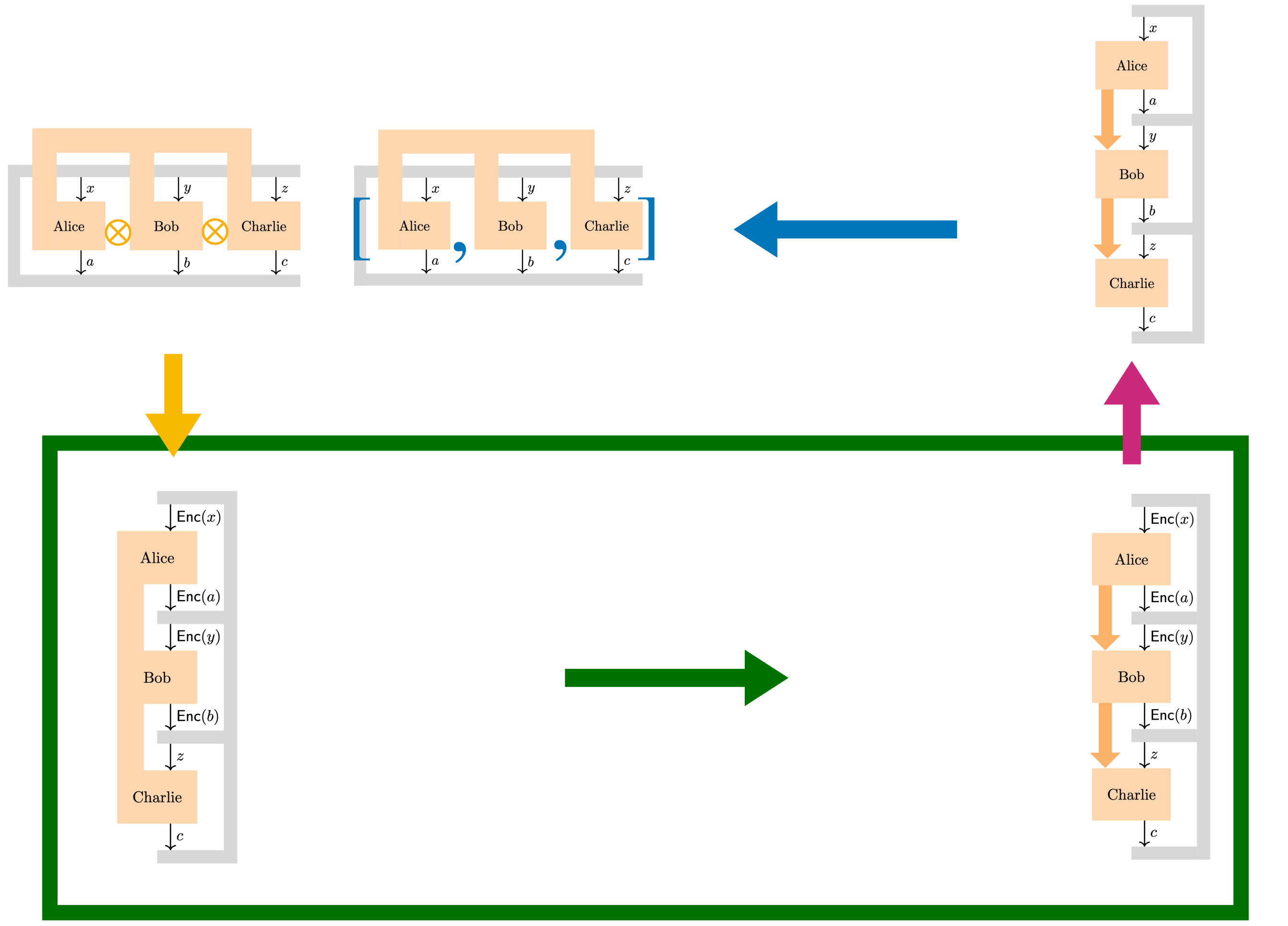

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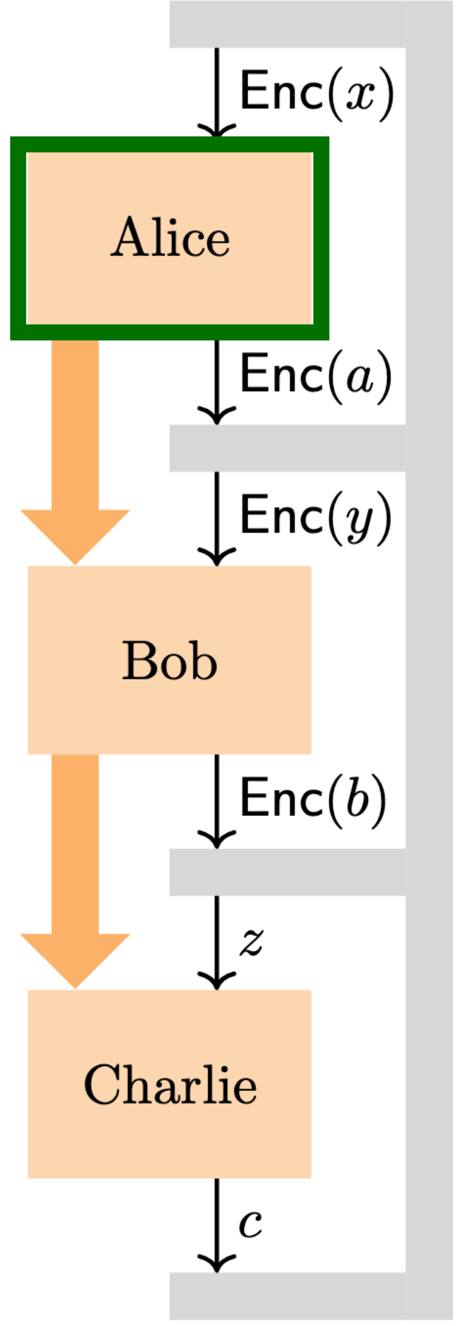

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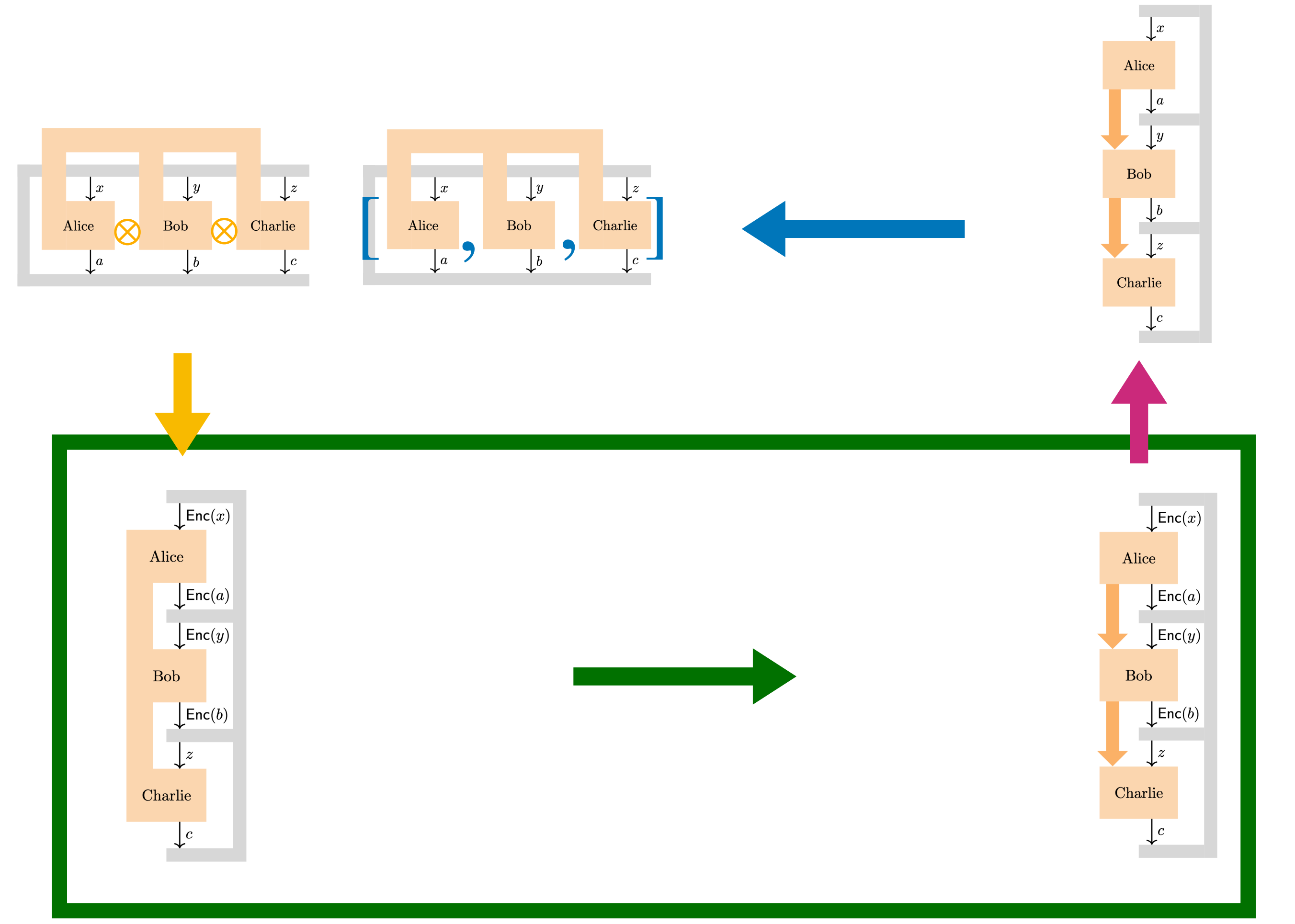

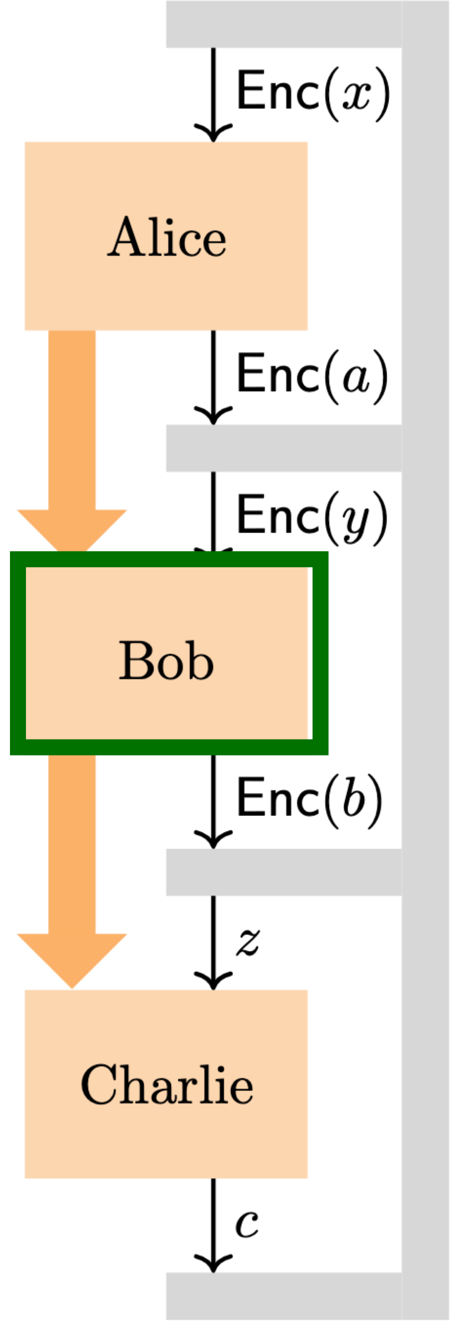

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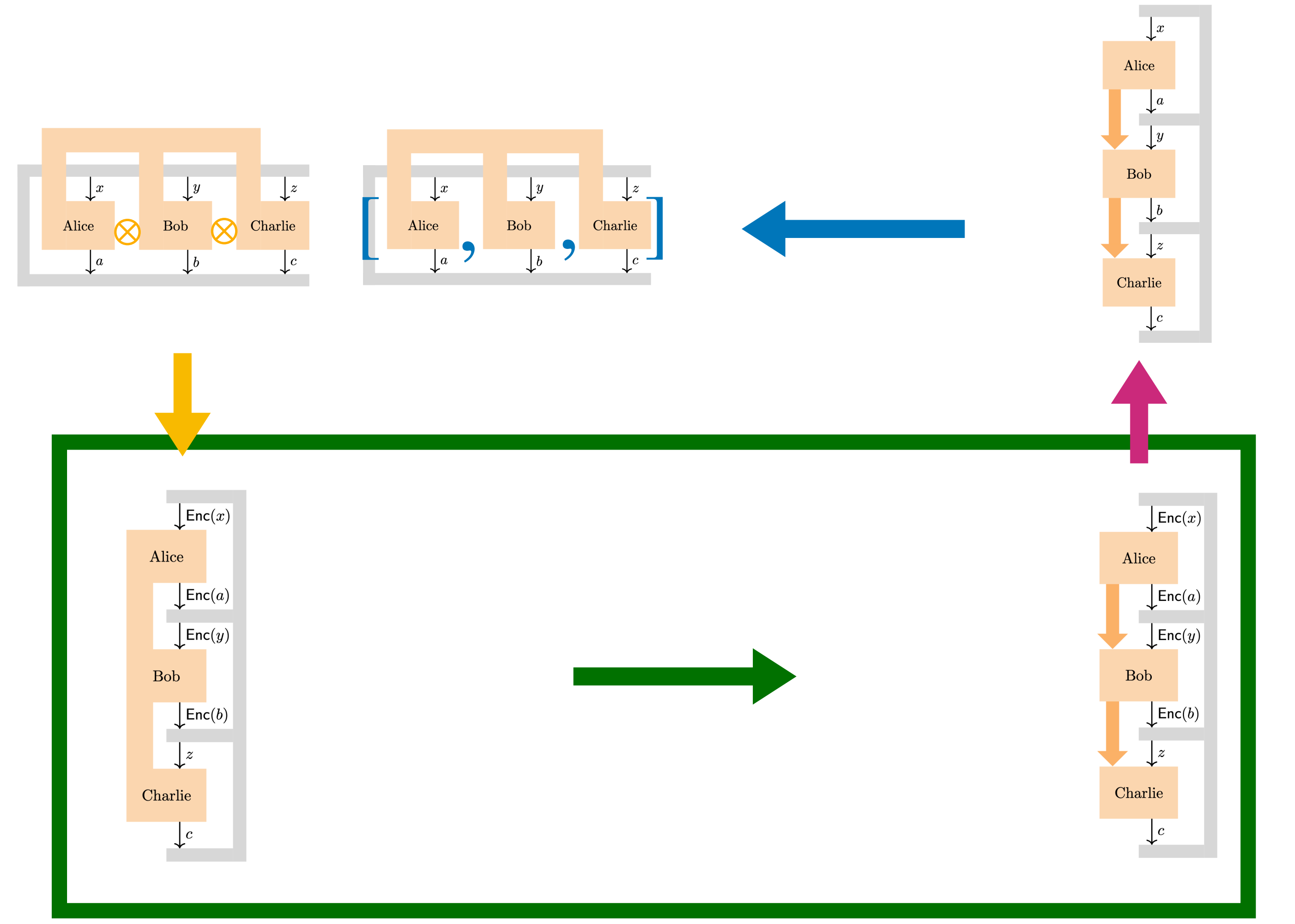

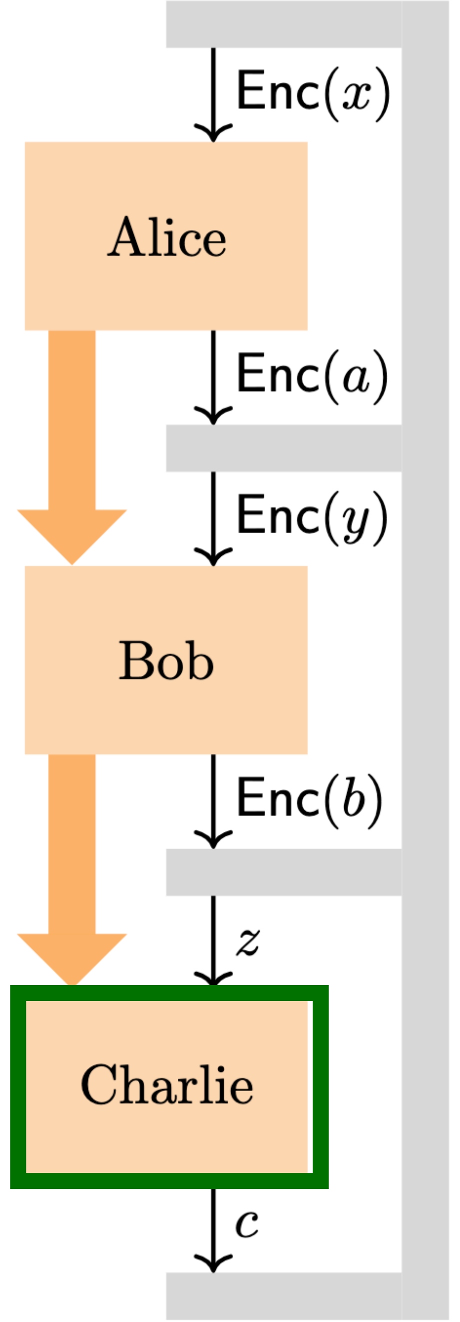

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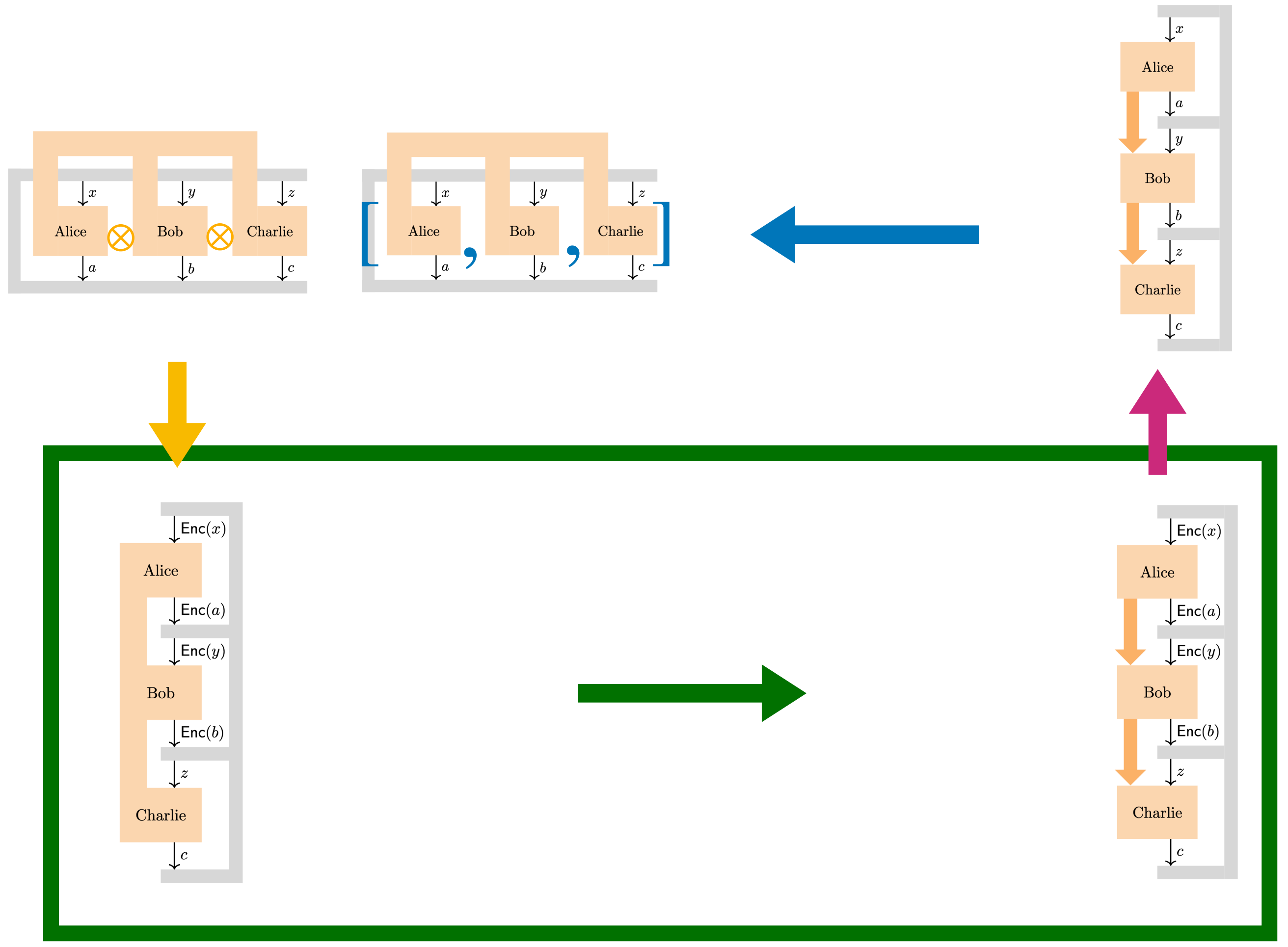

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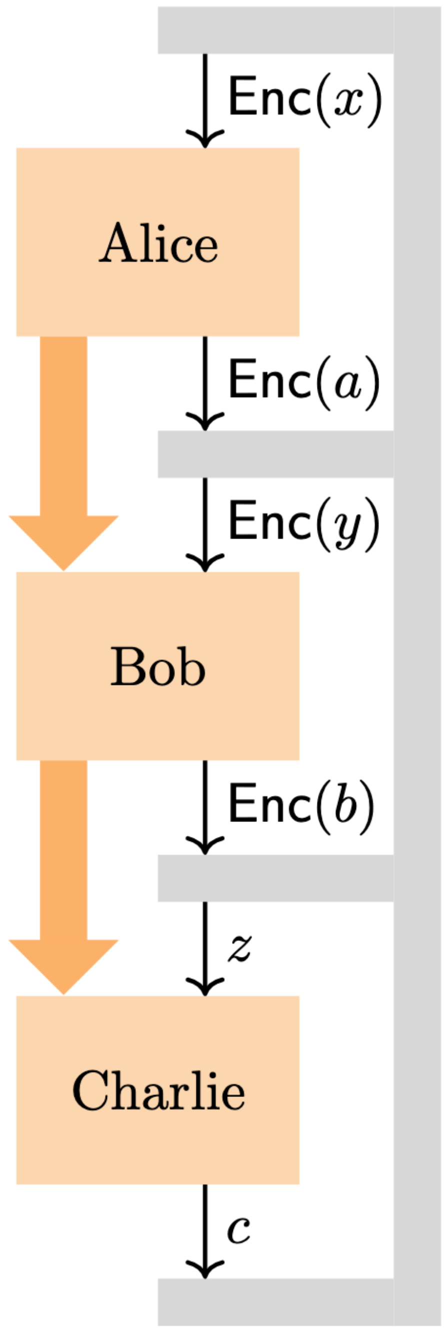

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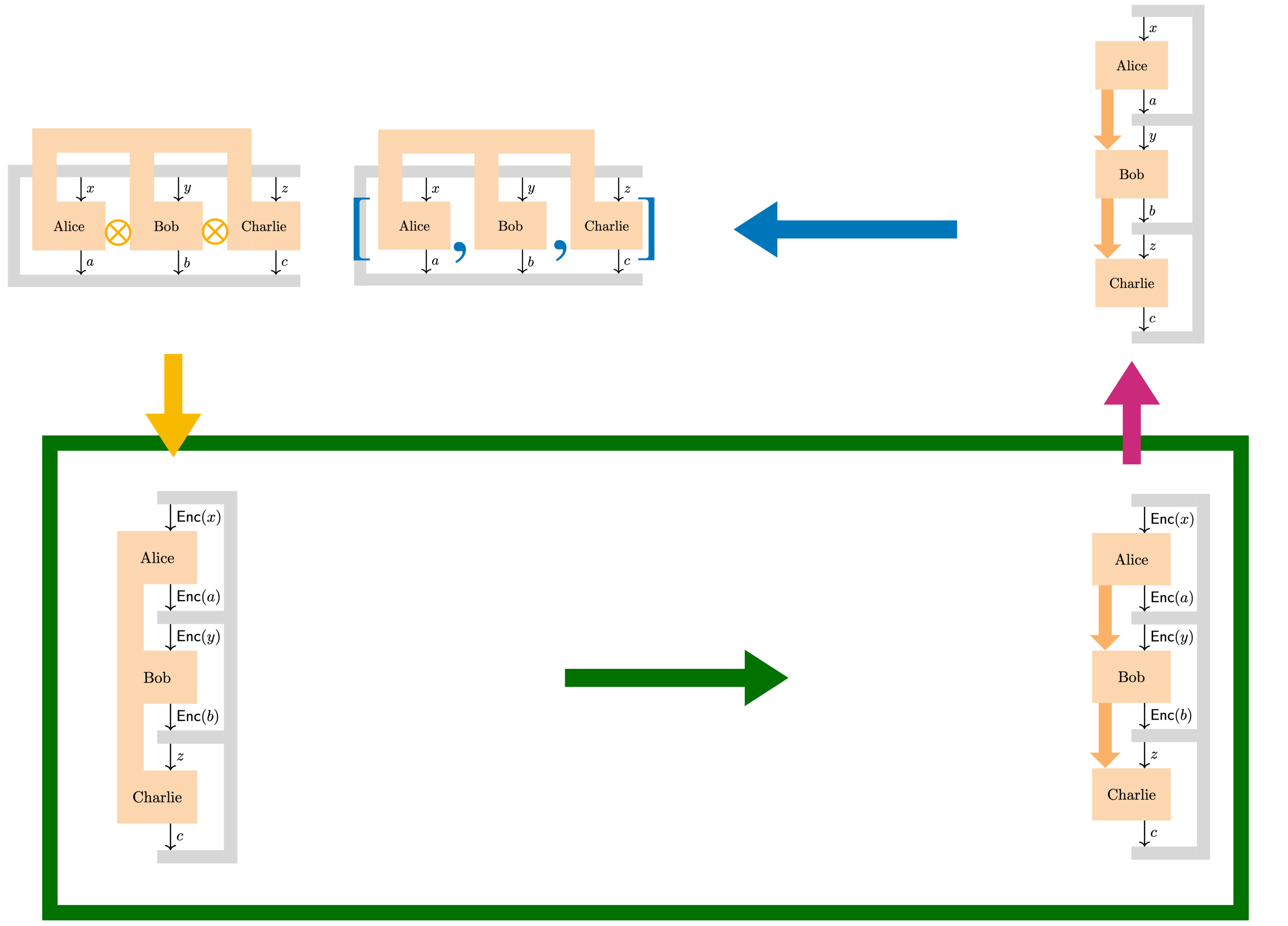

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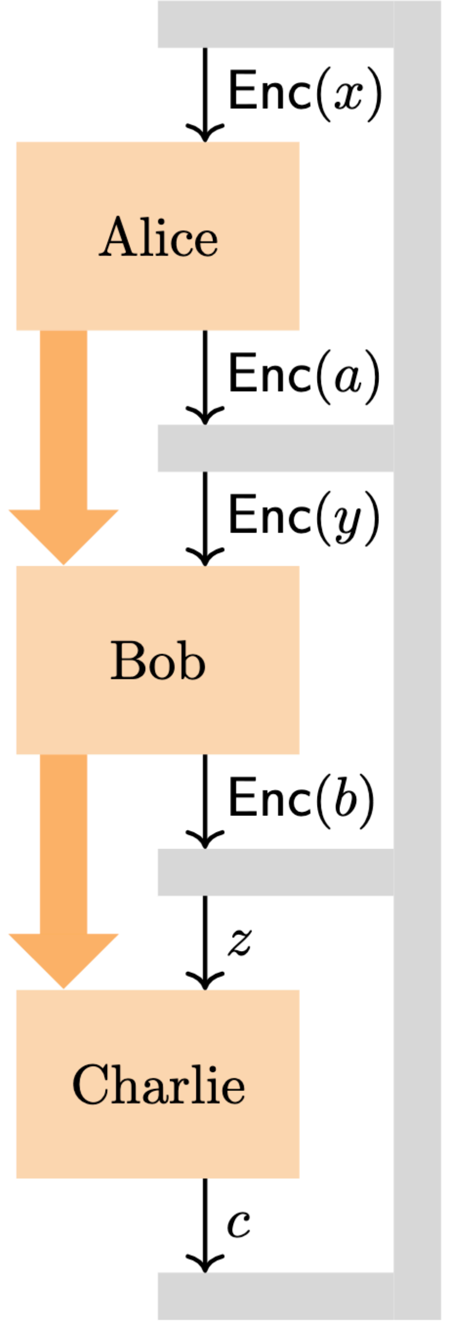

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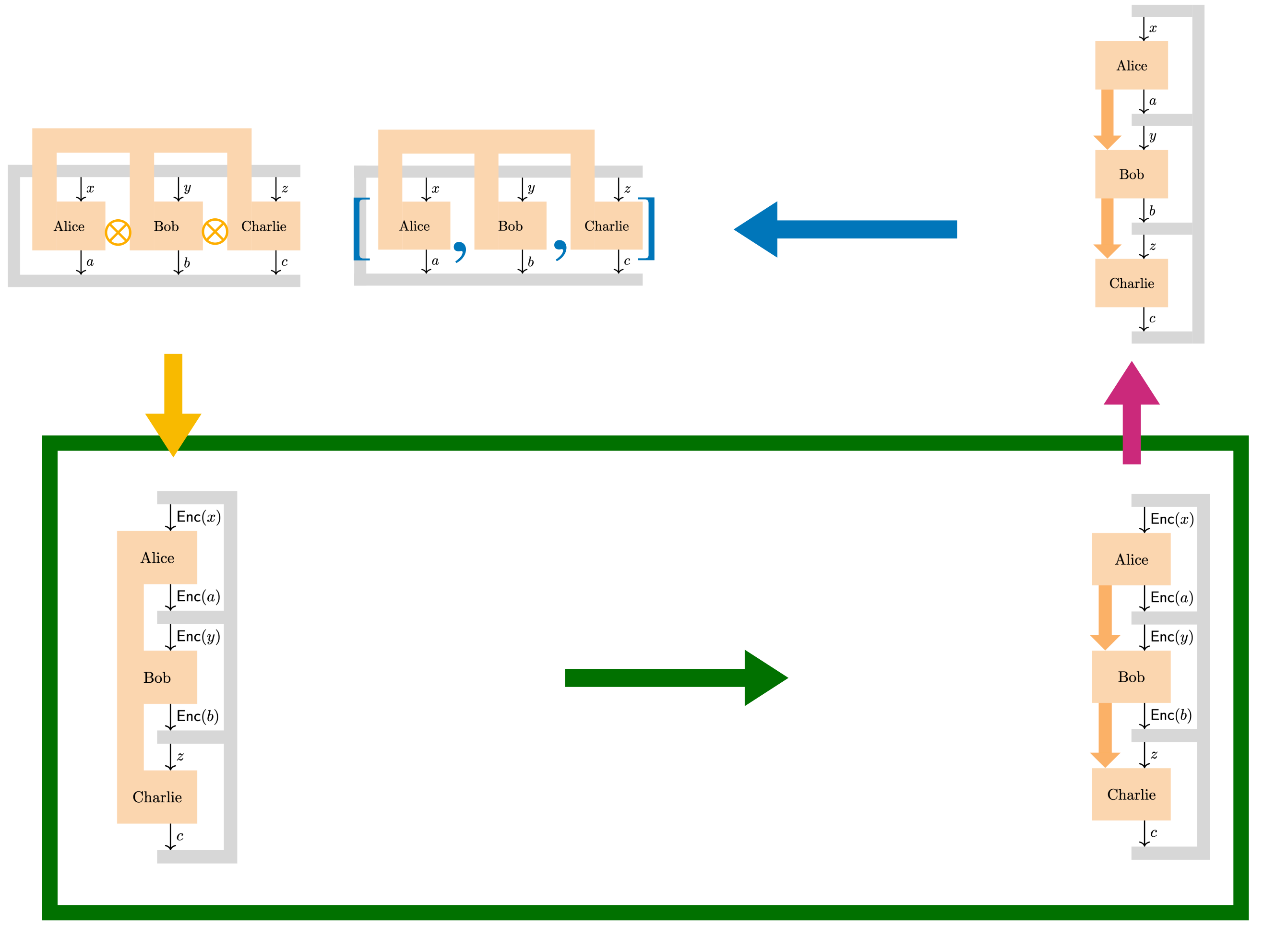

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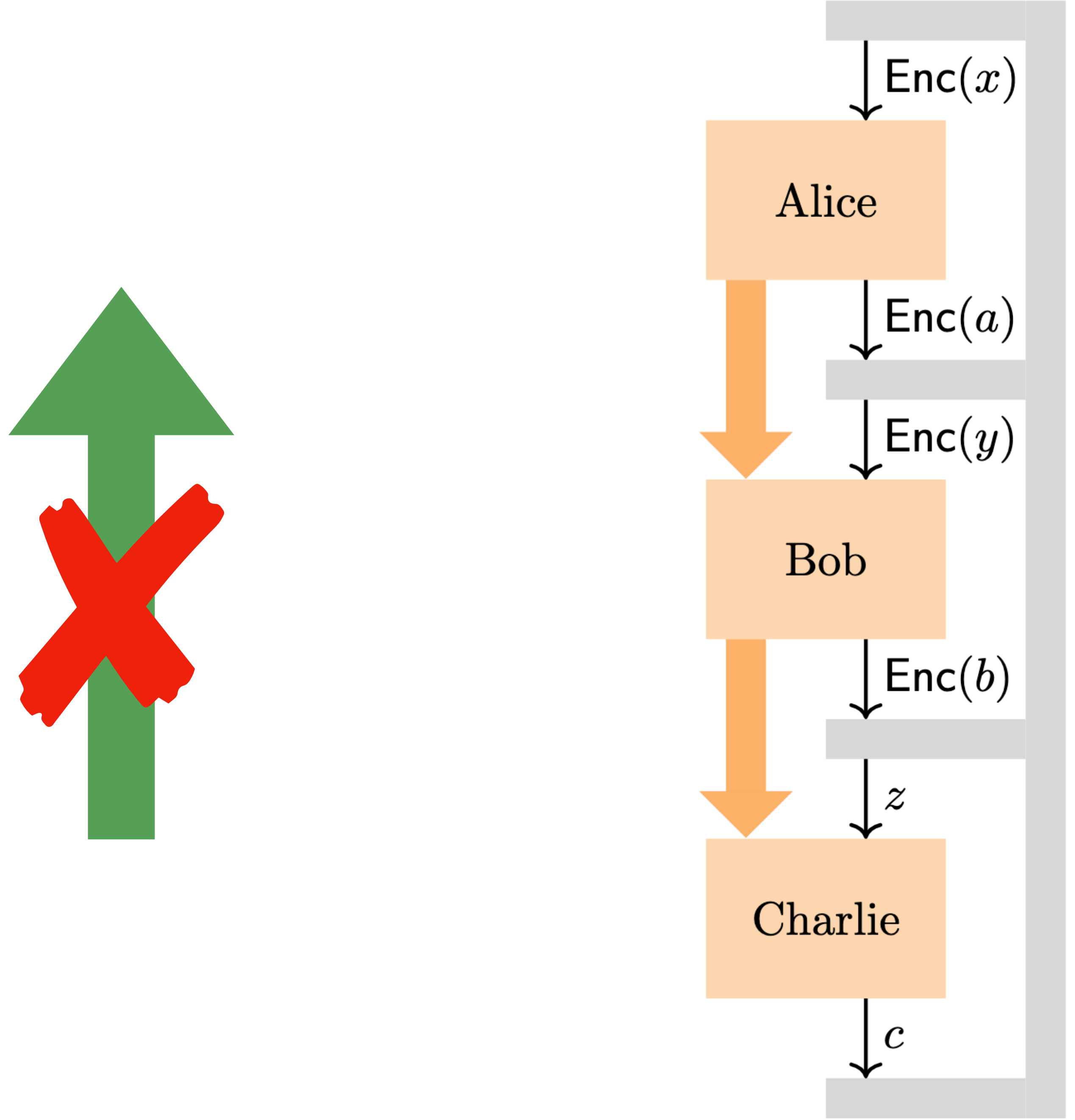

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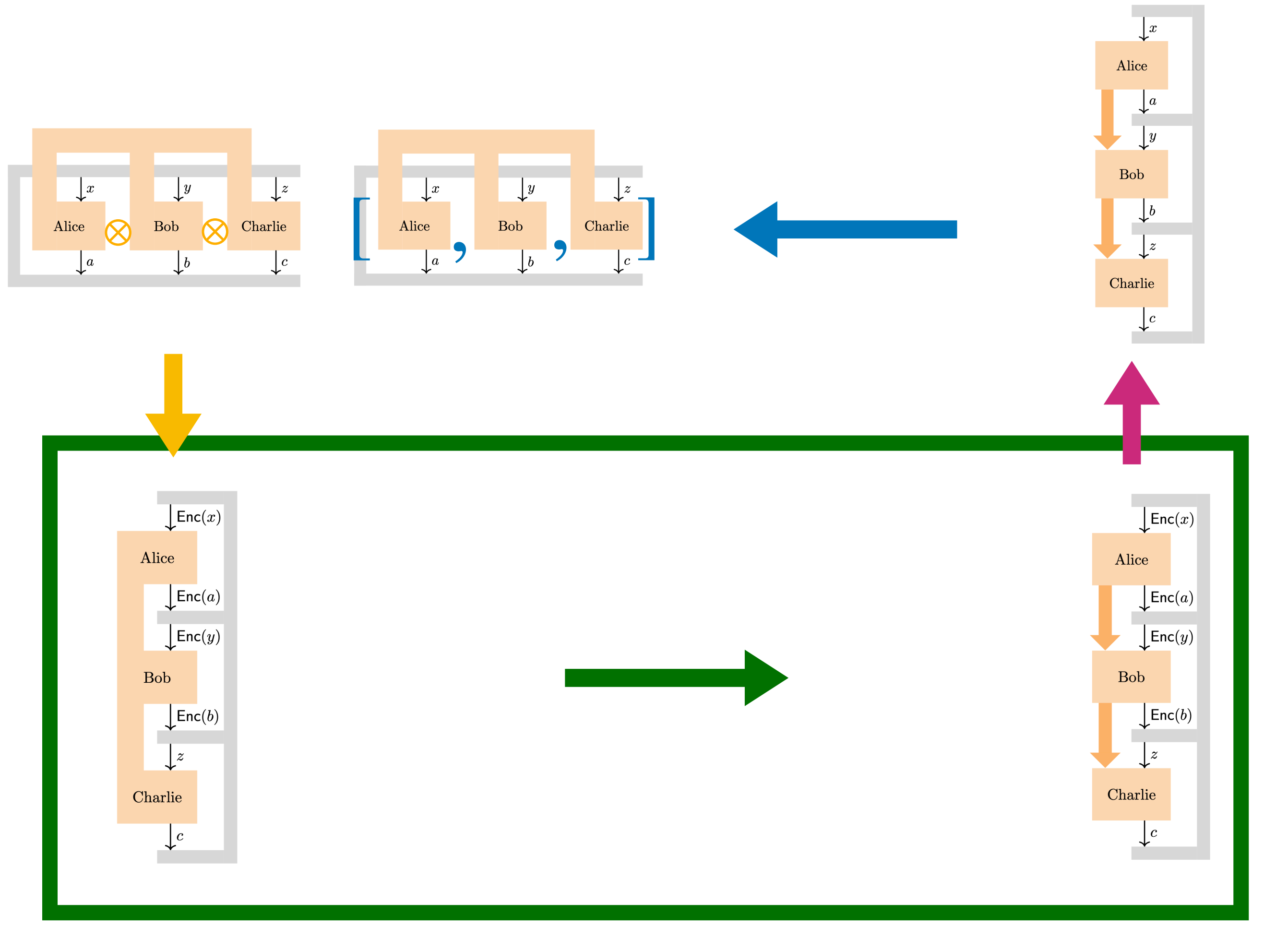

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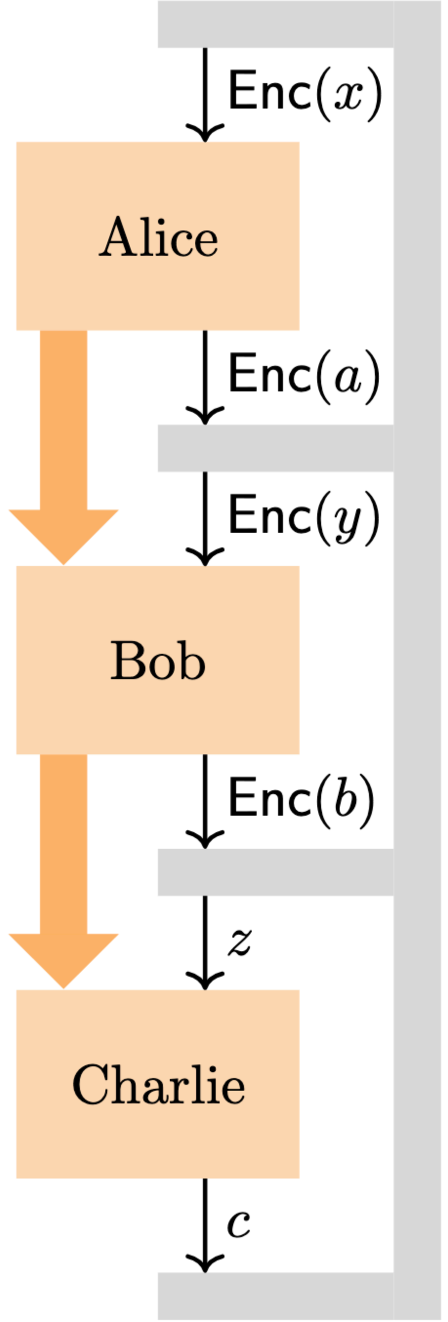
$$\sum_b : \quad \left|\mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{y}(\rho^\lambda_{a|x})\right] - \mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{y'}(\rho^\lambda_{a|x})\right]\right| \leq \mathsf{negl}(\lambda)$$

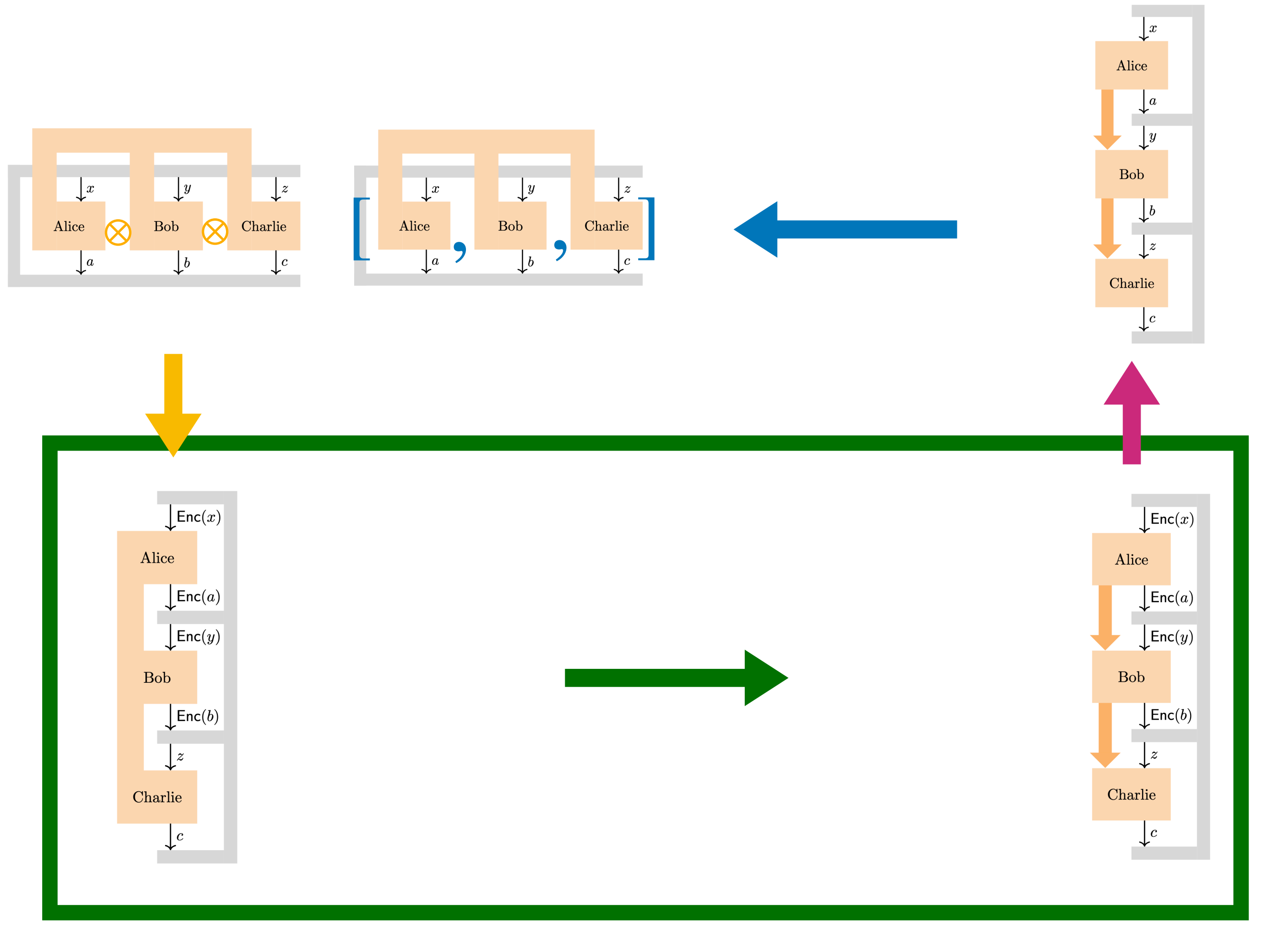
$$\sum_a : \quad \left|\mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{b|y}(\rho^\lambda_{x})\right] - \mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{b|y}(\rho^\lambda_{x'})\right]\right| \leq \mathsf{negl}(\lambda)$$



2. Constraints on the correlations

Quantum strategies

$$p_\lambda(a,b,c|x,y,z) = \mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{b|y}(\rho^\lambda_{a|x})\right]$$

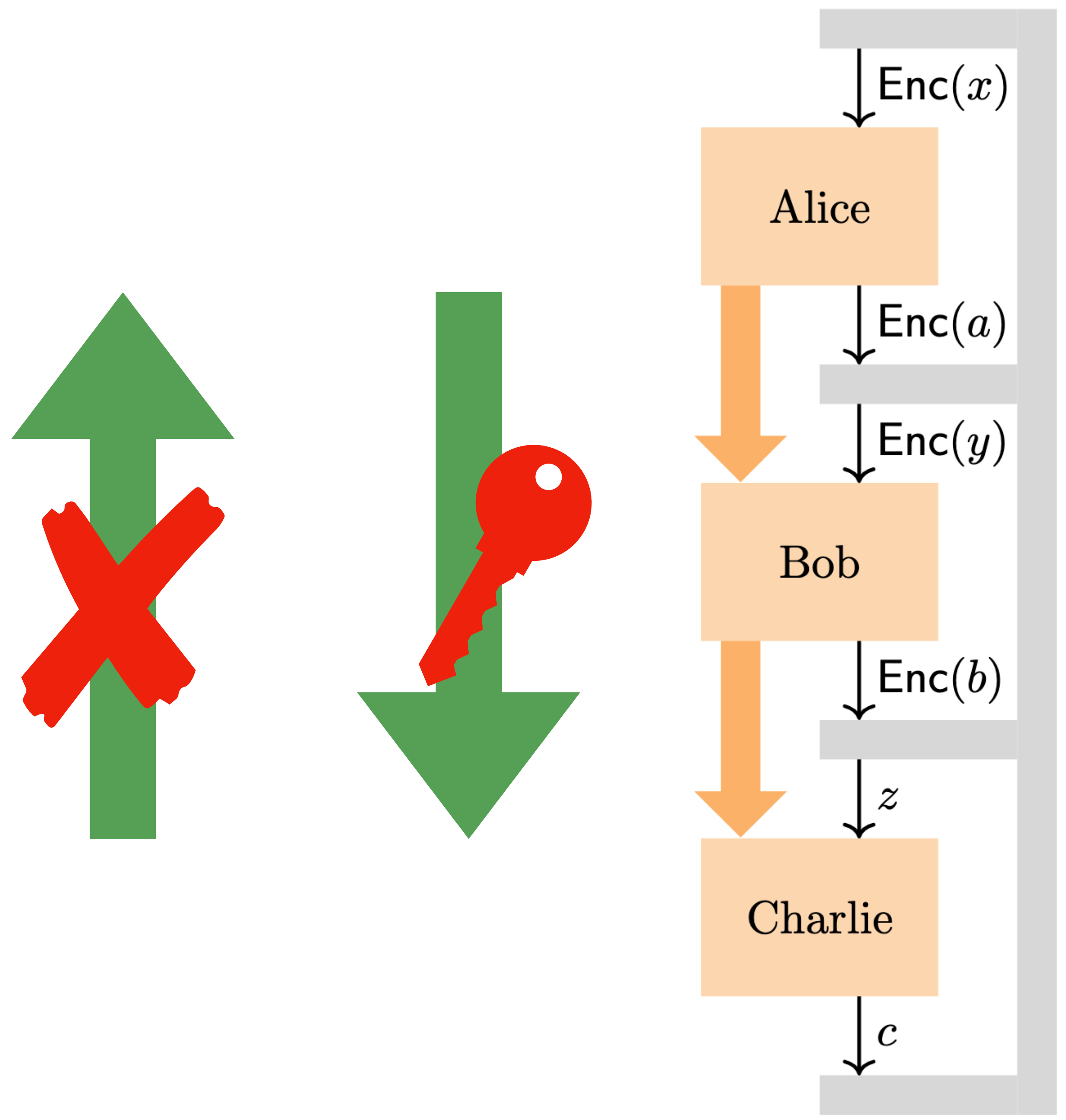


By construction: one-way almost non-signaling

$$\sum_c : \quad \left|\mathrm{Tr}\left[\mathbb{1}\tilde{B}^\lambda_{b|y}(\rho^\lambda_{a|x})\right] - \mathrm{Tr}\left[\mathbb{1}\tilde{B}^\lambda_{b|y}(\rho^\lambda_{a|x})\right]\right| = 0$$

$$\sum_b : \quad \left|\mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{y}(\rho^\lambda_{a|x})\right] - \mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{y'}(\rho^\lambda_{a|x})\right]\right| \leq \mathsf{negl}(\lambda)$$

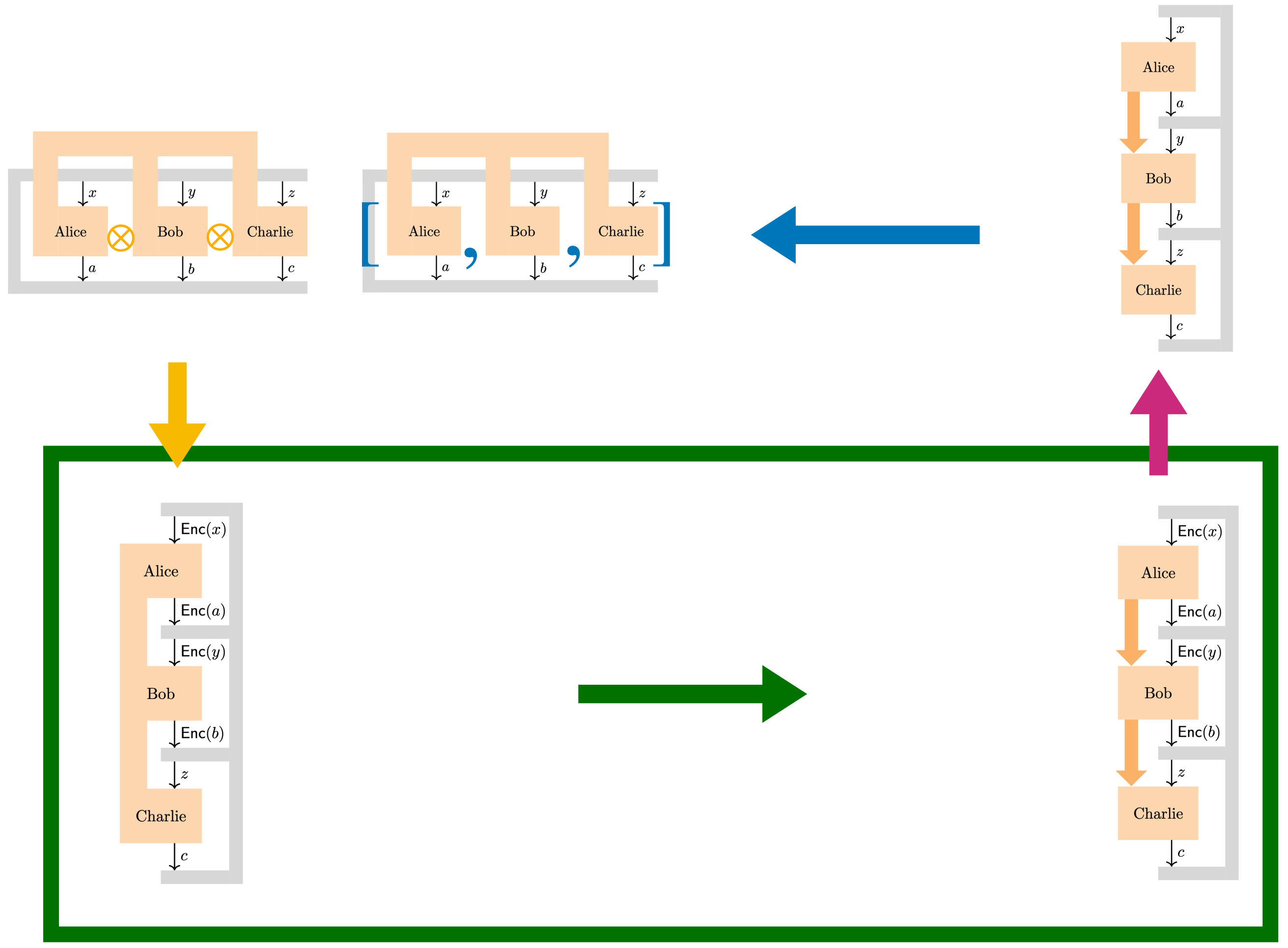
$$\sum_a : \quad \left|\mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{b|y}(\rho^\lambda_{x})\right] - \mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{b|y}(\rho^\lambda_{x'})\right]\right| \leq \mathsf{negl}(\lambda)$$



2. Constraints on the correlations

Block Encodings and efficient estimations

Stronger constraints from IND-CPA

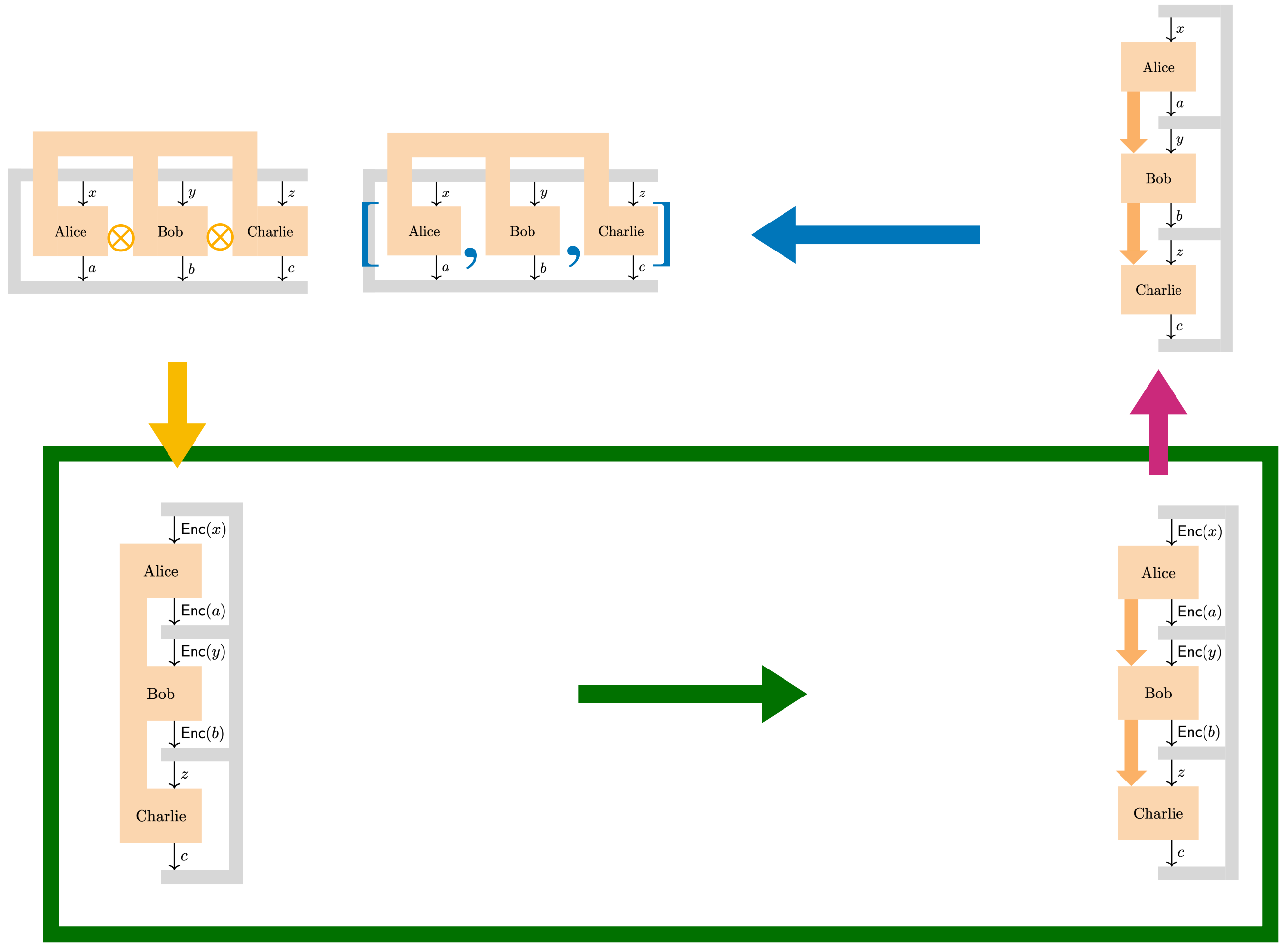


$$\left|\mathrm{Tr}\left[B^{\lambda}_{b|y}\rho^{\lambda}_{x}\right]-\mathrm{Tr}\left[B^{\lambda}_{b|y}\rho^{\lambda}_{x'}\right]\right|\leqslant \mathsf{negl}(\lambda)$$

$$\left|\mathrm{Tr}\left[P(\{B^{\lambda}_{b|y}\})\rho^{\lambda}_{x}\right]-\mathrm{Tr}\left[P(\{B^{\lambda}_{b|y}\})\rho^{\lambda}_{x'}\right]\right|\leqslant \mathit{negl}(\lambda)$$

2. Constraints on the correlations

Block Encodings and efficient estimations

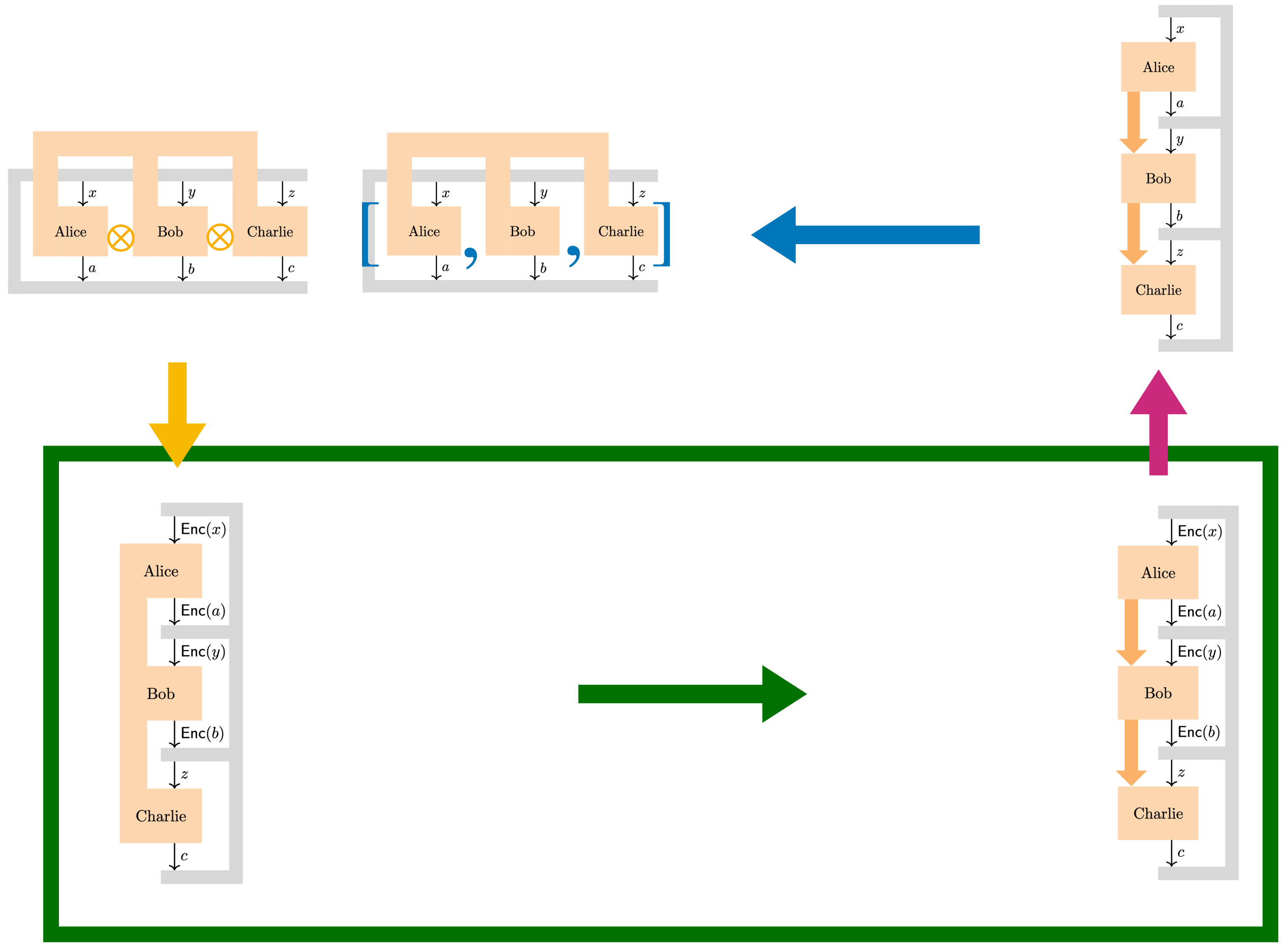


Stronger constraints from IND-CPA

$$\left|\mathrm{Tr}\left[B_{b|y}^{\lambda}\rho_{x}^{\lambda}\right]-\mathrm{Tr}\left[B_{b|y}^{\lambda}\rho_{x'}^{\lambda}\right]\right|\leqslant \mathsf{negl}(\lambda)$$

$$\rho_x^\lambda \approx_\lambda \rho_{x'}^\lambda$$

2. Constraints on the correlations



Block Encodings and efficient estimations

Stronger constraints from IND-CPA

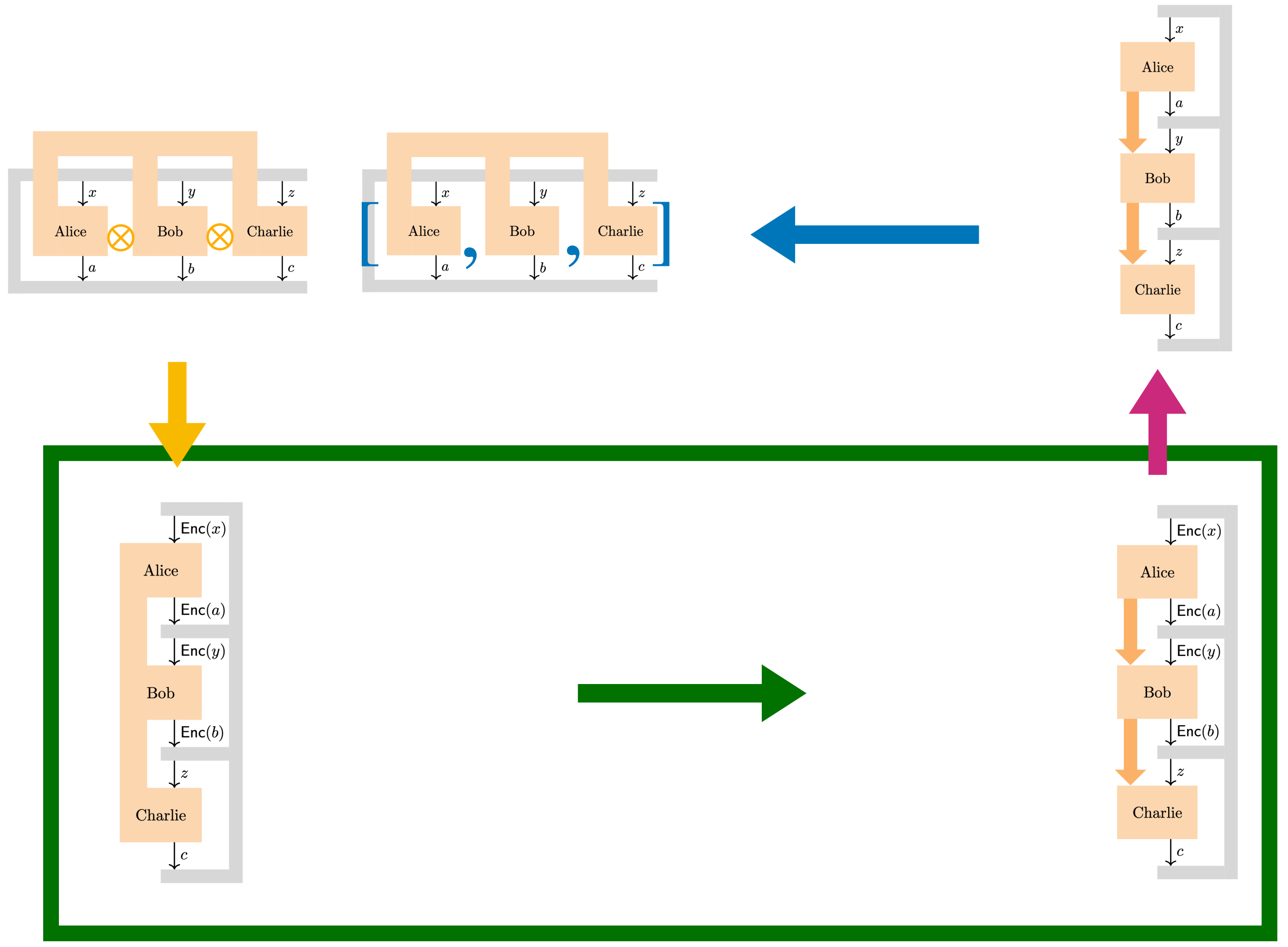
$$\left|\mathrm{Tr}\left[B^{\lambda}_{b|y}\rho^{\lambda}_{x}\right] - \mathrm{Tr}\left[B^{\lambda}_{b|y}\rho^{\lambda}_{x'}\right]\right| \leqslant \mathsf{negl}(\lambda)$$

$$\rho^{\lambda}_{x} \approx_{\lambda} \rho^{\lambda}_{x'}$$

$$\left|\mathrm{Tr}\left[C^{\lambda}_{c|z}\tilde{B}^{\lambda}_{y}(\rho^{\lambda}_{a|x})\right] - \mathrm{Tr}\left[C^{\lambda}_{c|z}\tilde{B}^{\lambda}_{y'}(\rho^{\lambda}_{a|x})\right]\right| \leq \mathsf{negl}(\lambda)$$

$$\left|\mathrm{Tr}\left[P(\{C^{\lambda}_{c|z}\})\tilde{B}^{\lambda}_{y}\left(\mathcal{R}^{\lambda}\rho^{\lambda}_{a|x}\mathcal{L}^{\lambda,*}\right)\right] - \mathrm{Tr}\left[P(\{C^{\lambda}_{c|z}\})\tilde{B}^{\lambda}_{y'}\left(\mathcal{R}^{\lambda}\rho^{\lambda}_{a|x}\mathcal{L}^{\lambda,*}\right)\right]\right| \leq \mathsf{negl}(\lambda)$$

2. Constraints on the correlations



Block Encodings and efficient estimations

Stronger constraints from IND-CPA

$$\left|\mathrm{Tr}\left[B^\lambda_{b|y}\rho^\lambda_x\right] - \mathrm{Tr}\left[B^\lambda_{b|y}\rho^\lambda_{x'}\right]\right| \leqslant \mathsf{negl}(\lambda)$$

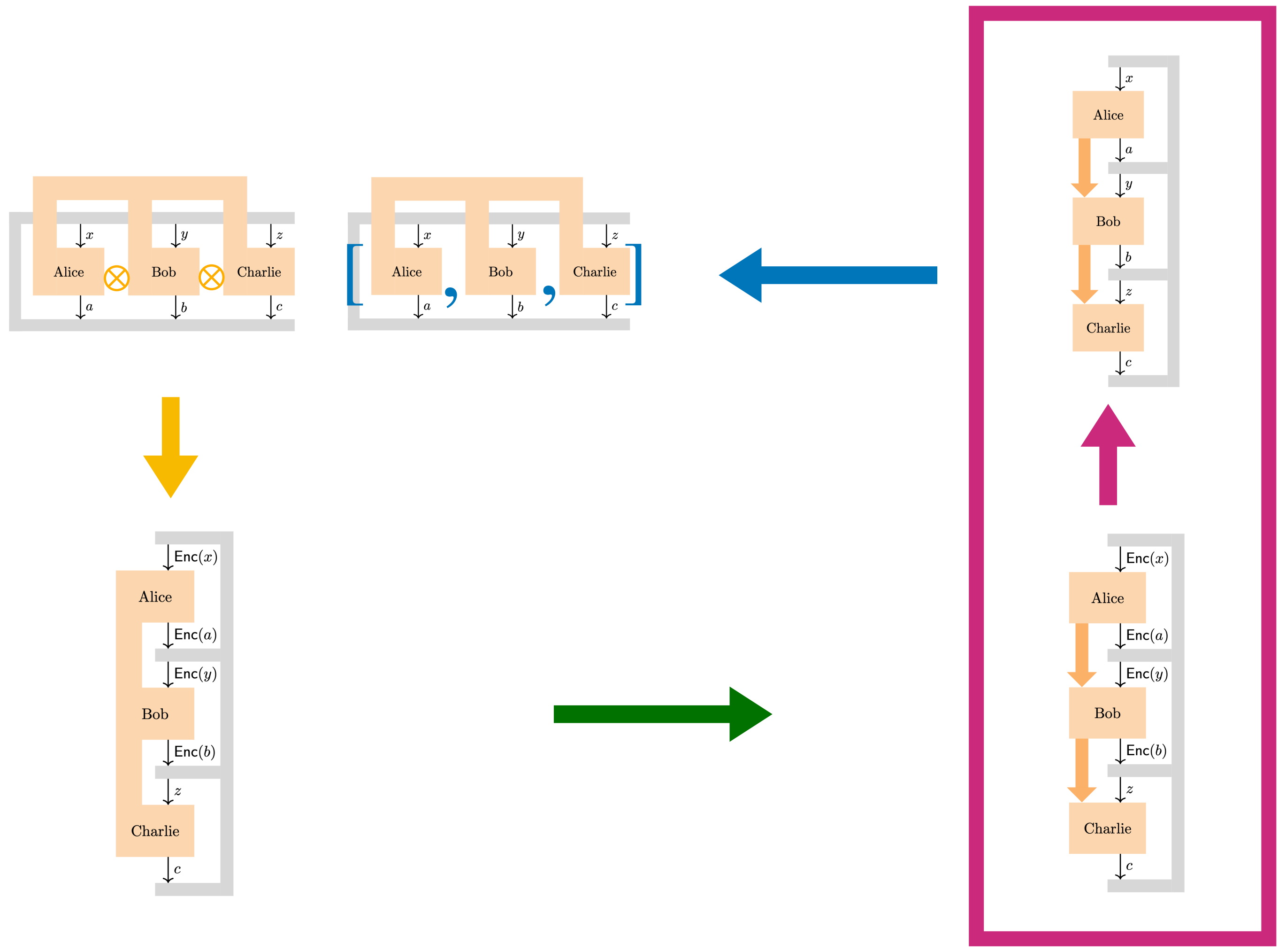
$$\boxed{\rho^\lambda_x \approx_\lambda \rho^\lambda_{x'}}$$

$$\left|\mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_y(\rho^\lambda_{a|x})\right] - \mathrm{Tr}\left[C^\lambda_{c|z}\tilde{B}^\lambda_{y'}(\rho^\lambda_{a|x})\right]\right| \leq \mathsf{negl}(\lambda)$$

$$\boxed{\tilde{B}^\lambda_y(\mathcal{R}^\lambda\rho^\lambda_{a|x}\mathcal{L}^{\lambda,*}) \approx_\lambda \tilde{B}^\lambda_{y'}(\mathcal{R}^\lambda\rho^\lambda_{a|x}\mathcal{L}^{\lambda,*})}$$

3. The asymptotic limit

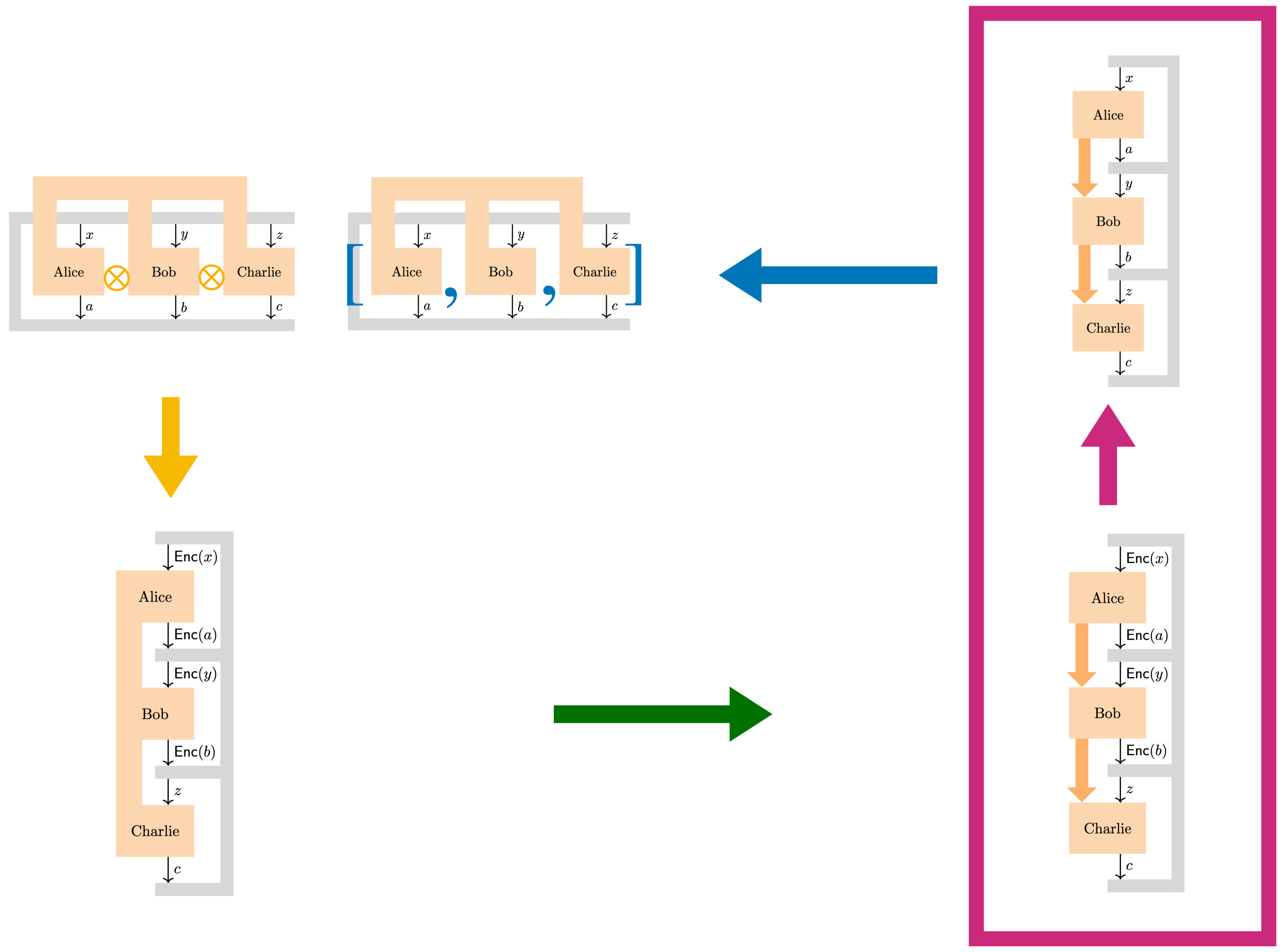
Algebraic strategies



Space	Hilbert space $\mathcal{H}$
Measurements	$C_{c\|z} \in \mathsf{B}(\mathcal{H})$
States	$\rho_{a\|x} \in \mathsf{B}(\mathcal{H})$
Transformations	$\tilde{B}_{b\|y} \in \mathsf{CP}(\mathcal{H})$
Correlations	$\mathrm{Tr}(C_{c\|z}\tilde{B}_{b\|y}(\rho_{a\|x}))$

3. The asymptotic limit

Algebraic strategies

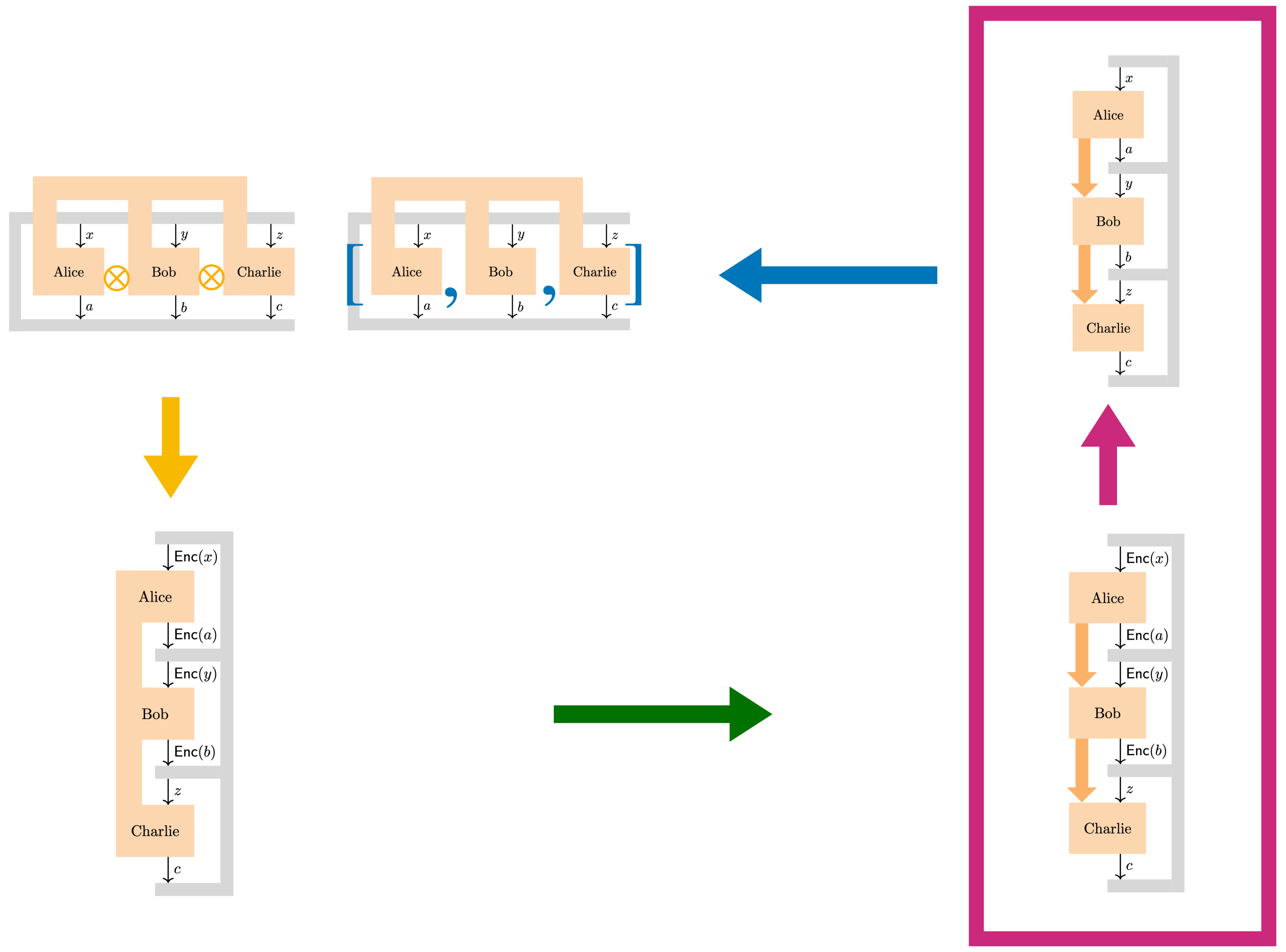


C*-algebras : algebraic & topological structure

Space	Hilbert space $\mathcal{H}$	C*-algebra $\mathscr{A}, \mathscr{B}$
Measurements	$C_{c\|z} \in \mathsf{B}(\mathcal{H})$	$\mathfrak{m}_{c\|z} \in \mathscr{A}$
States	$\rho_{a\|x} \in \mathsf{B}(\mathcal{H})$	$\phi_{a\|x} : \mathscr{B} \to \mathbb{C}$
Transformations	$\tilde{B}_{b\|y} \in \mathsf{CP}(\mathcal{H})$	$T_{b\|y} : \mathscr{A} \to \mathscr{B}$
Correlations	$\mathrm{Tr}(C_{c\|z}\tilde{B}_{b\|y}(\rho_{a\|x}))$	$\phi_{a\|x}(T_{b\|y}(\mathfrak{m}_{c\|z}))$

3. The asymptotic limit

Algebraic strategies

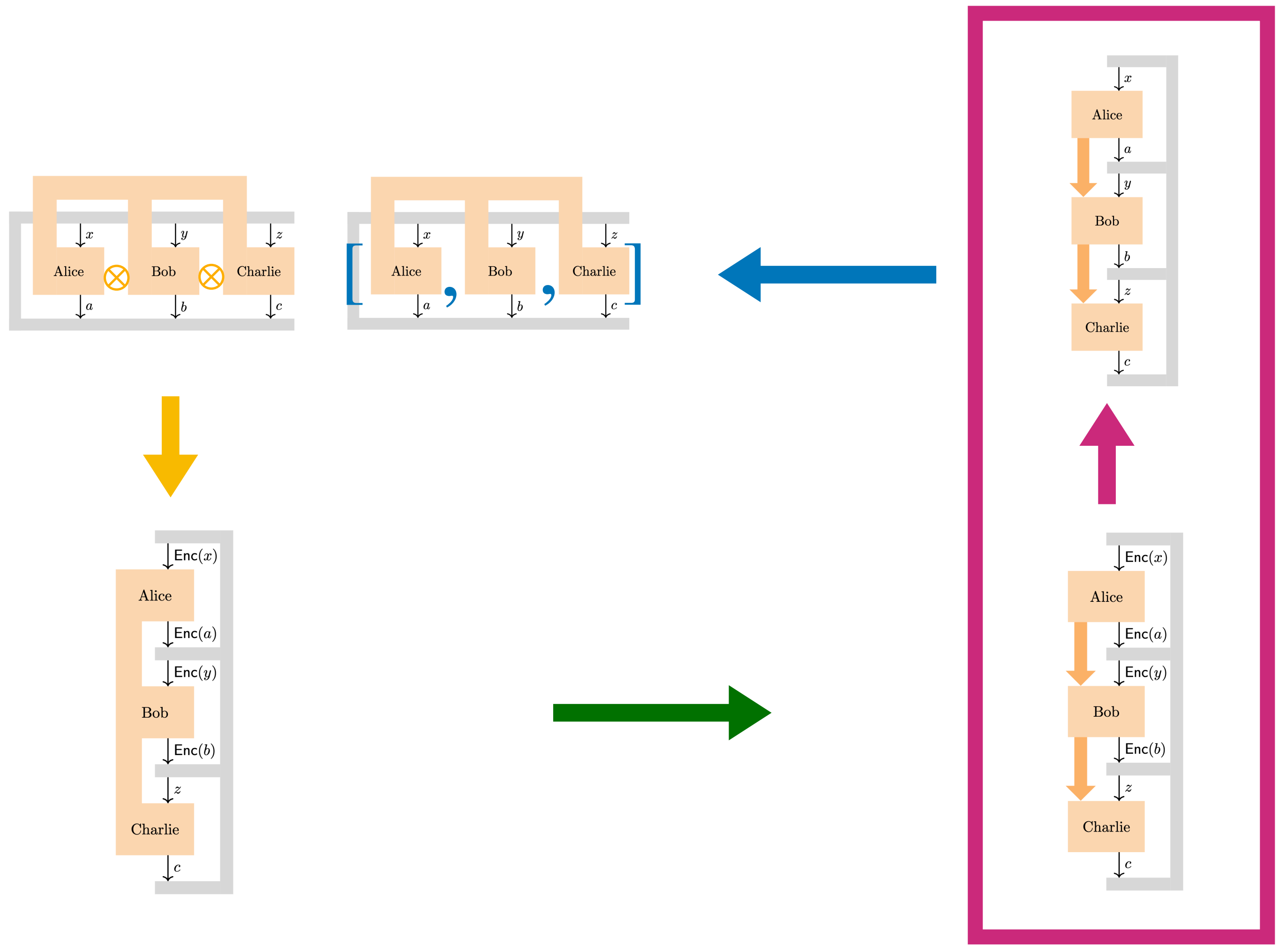


C*-algebras : algebraic & topological structure

Space	Hilbert space $\mathcal{H}$	C*-algebra $\mathscr{A}, \mathscr{B}$	
Measurements	$C_{c\|z} \in \mathsf{B}(\mathcal{H})$	$\mathfrak{m}_{c\|z} \in \mathscr{A}$	$\mathfrak{m}_{c\|z} \sim C_{c\|z}$
States	$\rho_{a\|x} \in \mathsf{B}(\mathcal{H})$	$\phi_{a\|x} : \mathscr{B} \to \mathbb{C}$	$\phi_{a\|x}(\cdot) \sim \mathrm{Tr}(\cdot \rho_{a\|x})$
Transformations	$\tilde{B}_{b\|y} \in \mathsf{CP}(\mathcal{H})$	$T_{b\|y} : \mathscr{A} \to \mathscr{B}$	$T_{b\|y} \sim \tilde{B}^*_{b\|y}$
Correlations	$\mathrm{Tr}(C_{c\|z} \tilde{B}_{b\|y}(\rho_{a\|x}))$	$\phi_{a\|x}(T_{b\|y}(\mathfrak{m}_{c\|z}))$	

3. The asymptotic limit

Algebraic strategies



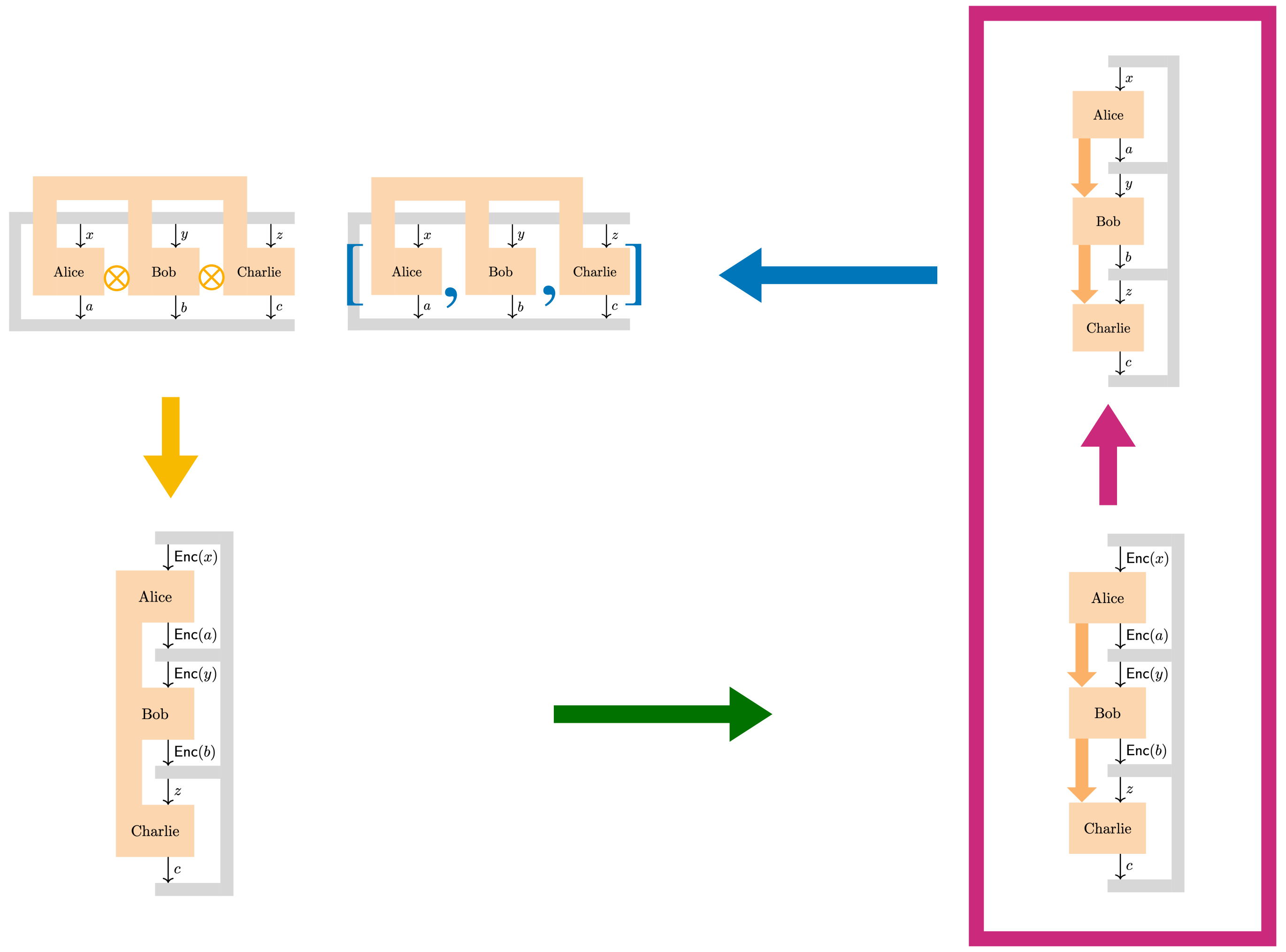
C*-algebras : algebraic & topological structure

Space	Hilbert space $\mathcal{H}^{\lambda}$	C*-algebra $\mathscr{A}, \mathscr{B}$	
Measurements	$C^{\lambda}_{c\|z} \in \mathsf{B}(\mathcal{H}^{\lambda})$	$\mathfrak{m}_{c\|z} \in \mathscr{A}$	$\mathfrak{m}_{c\|z} \sim C_{c\|z}$
States	$\rho^{\lambda}_{a\|x} \in \mathsf{B}(\mathcal{H}^{\lambda})$	$\phi^{\lambda}_{a\|x} : \mathscr{B} \to \mathbb{C}$	$\phi_{a\|x}(\cdot) \sim \mathrm{Tr}(\cdot \rho_{a\|x})$
Transformations	$\tilde{B}^{\lambda}_{b\|y} \in \mathsf{CP}(\mathcal{H}^{\lambda})$	$T_{b\|y} : \mathscr{A} \to \mathscr{B}$	$T_{b\|y} \sim \tilde{B}^{*}_{b\|y}$
Correlations	$\mathrm{Tr}(C^{\lambda}_{c\|z} \tilde{B}^{\lambda}_{b\|y}(\rho^{\lambda}_{a\|x}))$	$\phi^{\lambda}_{a\|x}(T_{b\|y}(\mathfrak{m}_{c\|z}))$	

3. The asymptotic limit

Universal C* algebras of PVMs

2 players



$$\begin{tikzcd} \mathscr{A}_B \arrow[d, "\theta^\lambda"'] \\ \mathrm{B}(\mathcal{H}^\lambda) \end{tikzcd}$$

Universal C* algebras of PVMs

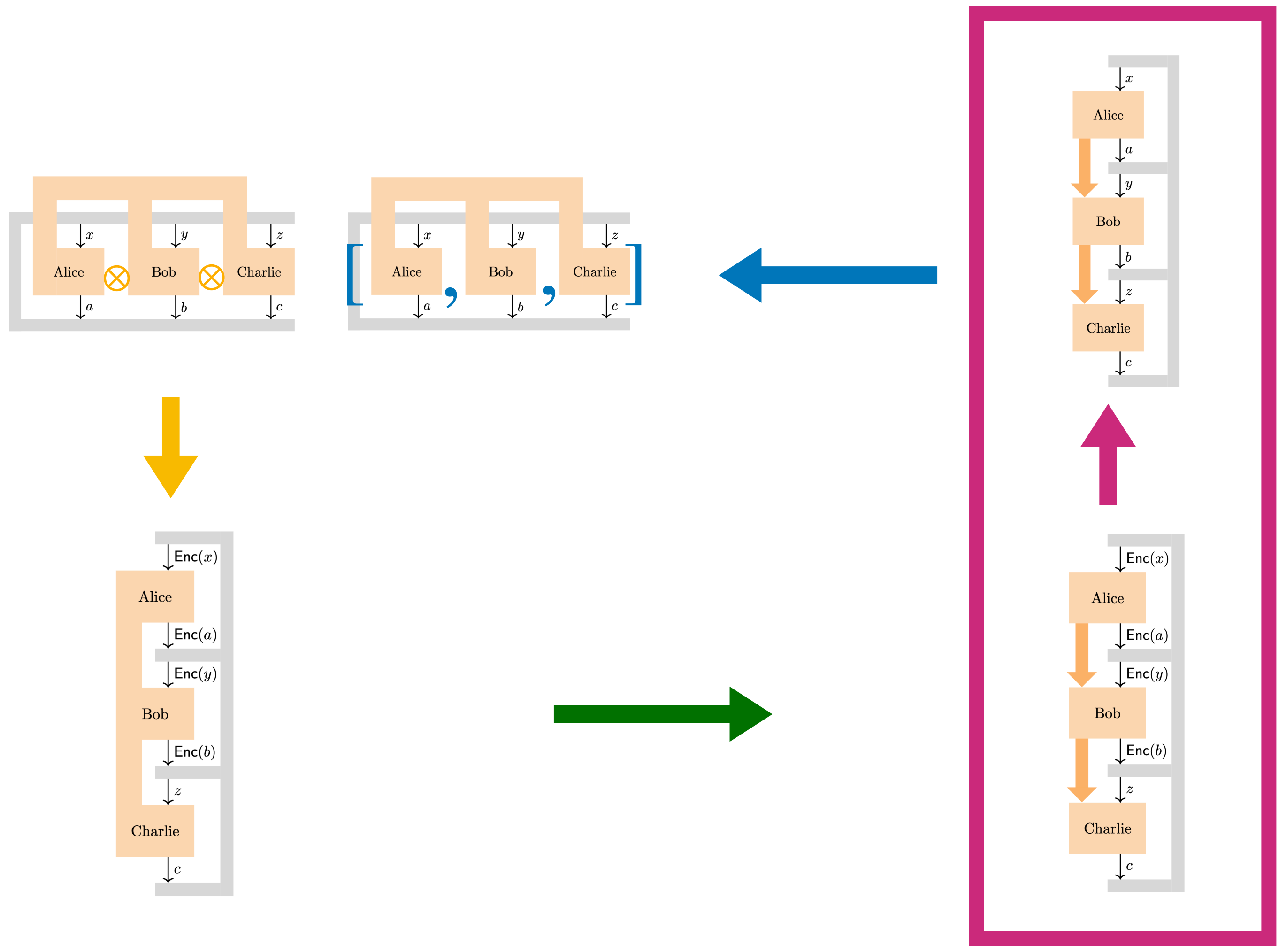
Generated by $e_{b|y} \in \mathscr{A}_B$ s.t.

1. $e_{b|y} = e^*_{b|y}$
2. $0 \leq e_{b|y} \leq 1$
3. $\sum_b e_{b|y} = 1$
4. $e_{b|y} e_{b'|y} = \delta_{b,b'} e_{b|y}$

3. The asymptotic limit

Universal C* algebras of PVMs

2 players



$$\begin{tikzcd} \mathscr{A}_B \arrow[d, "\theta^\lambda"'] \arrow[dr, "\phi^\lambda_{a|x}"] & \\ \mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "\mathrm{Tr}(\sigma^\lambda_{a|x}\,\cdot)"'] & \mathbb{C} \end{tikzcd}$$

Universal C* algebras of PVMs

Generated by $e_{b|y} \in \mathscr{A}_B$ s.t.

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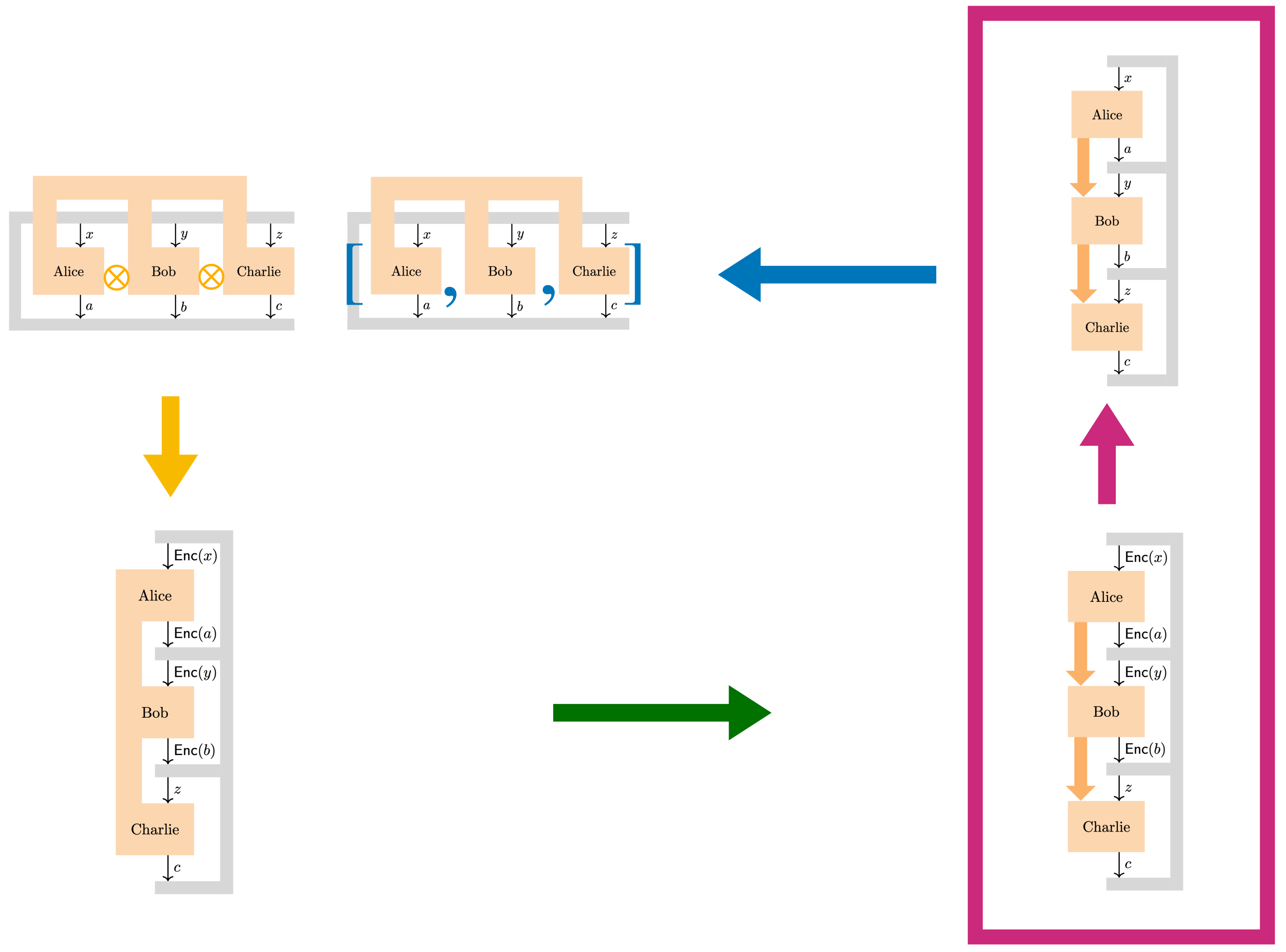
4. $e_{b|y} e_{b'|y} = \delta_{b,b'} e_{b|y}$

3. The asymptotic limit

Universal C* algebras of PVMs

2 players



$$\begin{tikzcd} \mathscr{A}_B \arrow[d, "\theta^\lambda"'] \arrow[dr, "\phi^\lambda_{a|x}"] & \\ \mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "\mathrm{Tr}(\sigma^\lambda_{a|x}\,\cdot)"'] & \mathbb{C} \end{tikzcd}$$

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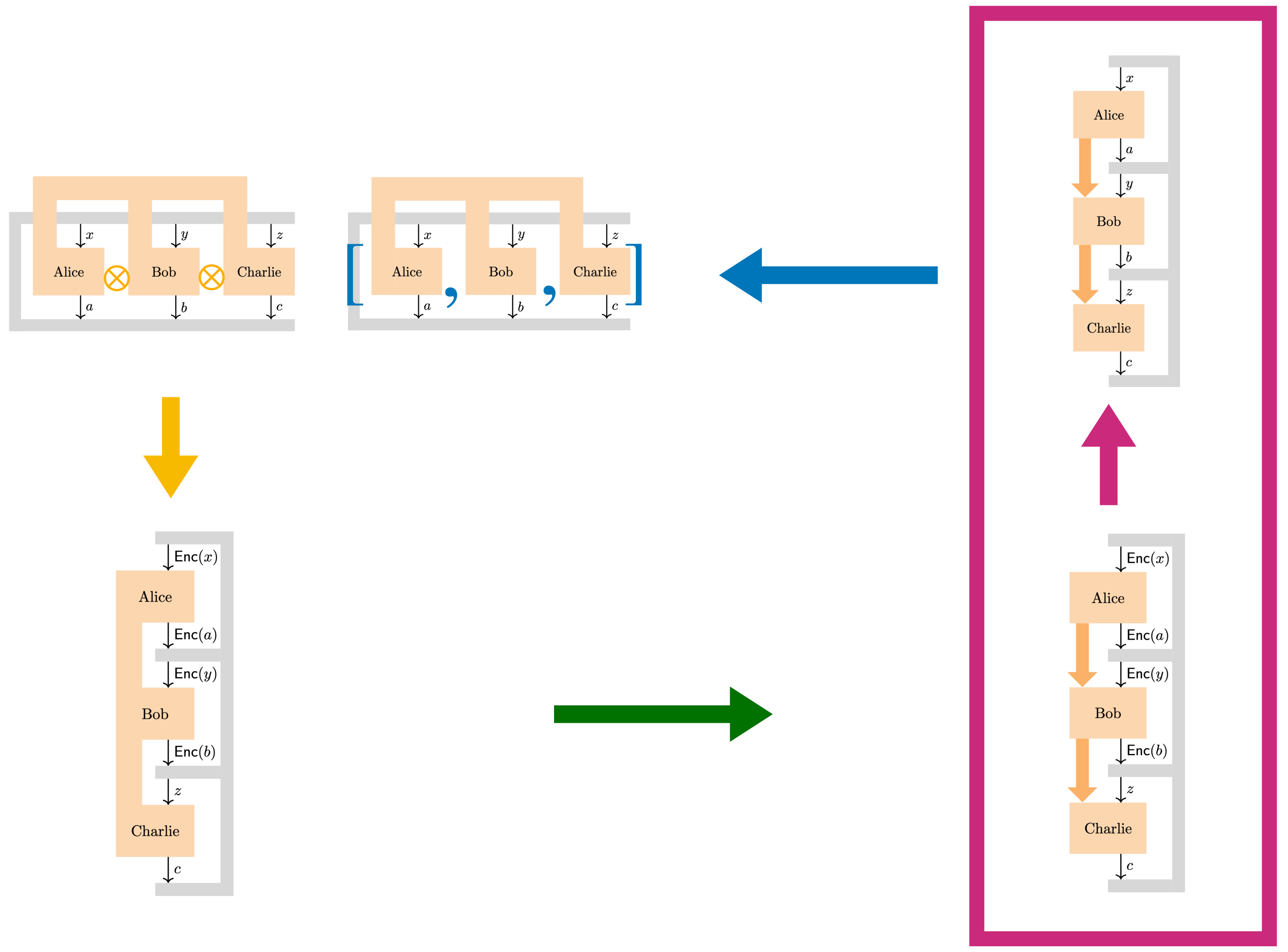
3. The asymptotic limit

Universal C* algebras of PVMs

2 players



$$
\begin{tikzcd}
\mathscr{A}_B \arrow[d, "\theta^\lambda"'] \arrow[dr, "\phi^\lambda_{a|x}"] & \\
\mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "\mathrm{Tr}(\sigma^\lambda_{a|x}\,\cdot)"'] & \mathbb{C}
\end{tikzcd}
\qquad \xrightarrow{\lambda\to\infty} \qquad \phi_{a|x}
$$

Universal C* algebras of PVMs

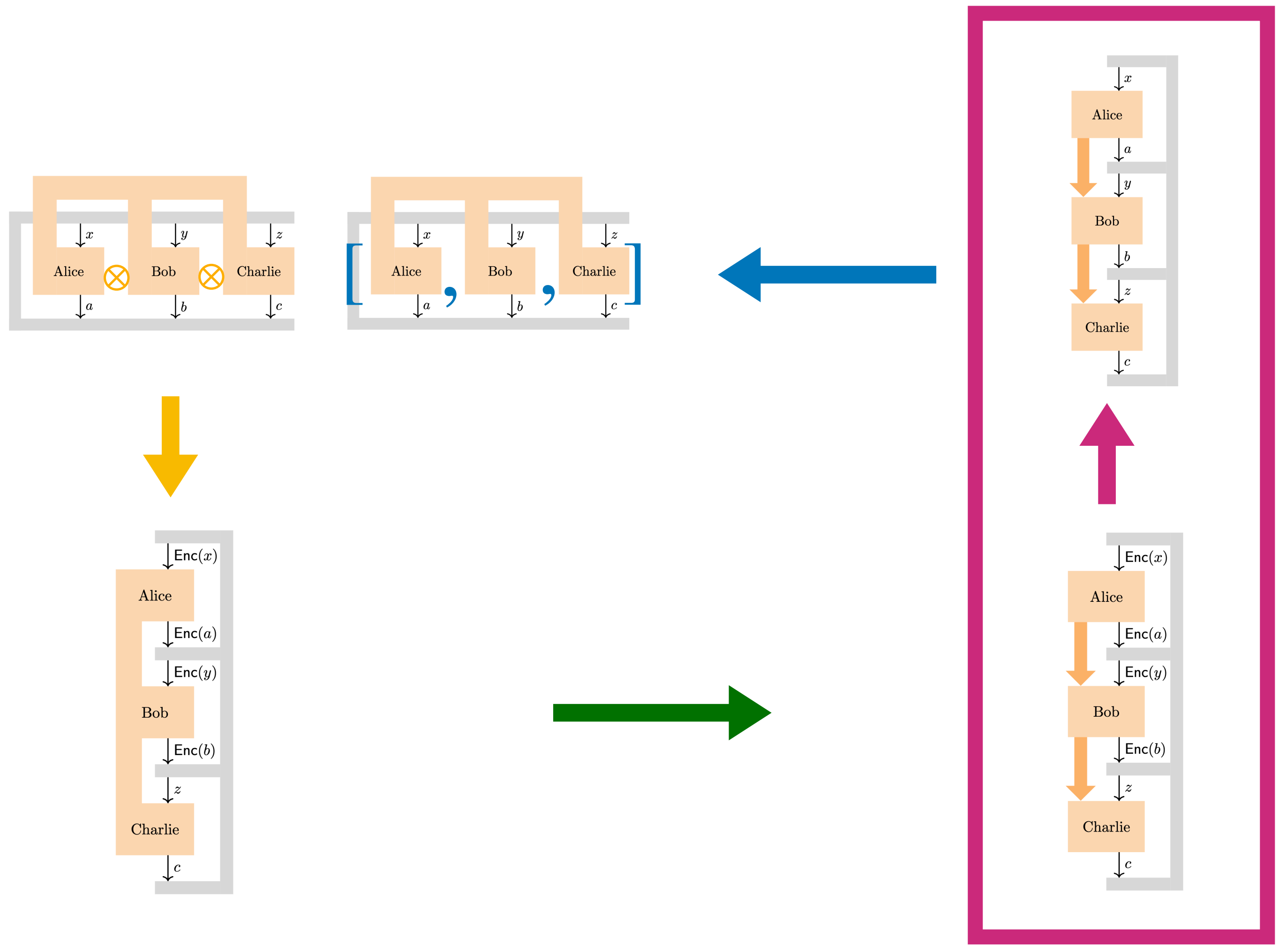
Generated by $e_{b|y} \in \mathscr{A}_B$ s.t.

1. $e_{b|y} = e^*_{b|y}$
2. $0 \leq e_{b|y} \leq 1$
3. $\sum_b e_{b|y} = 1$
4. $e_{b|y} e_{b'|y} = \delta_{b,b'} e_{b|y}$

3. The asymptotic limit

Universal C* algebras of PVMs

2 players



$$\phi_x^\lambda \approx_\lambda \phi_{x'}^\lambda$$

$$\phi_x = \phi_{x'}$$

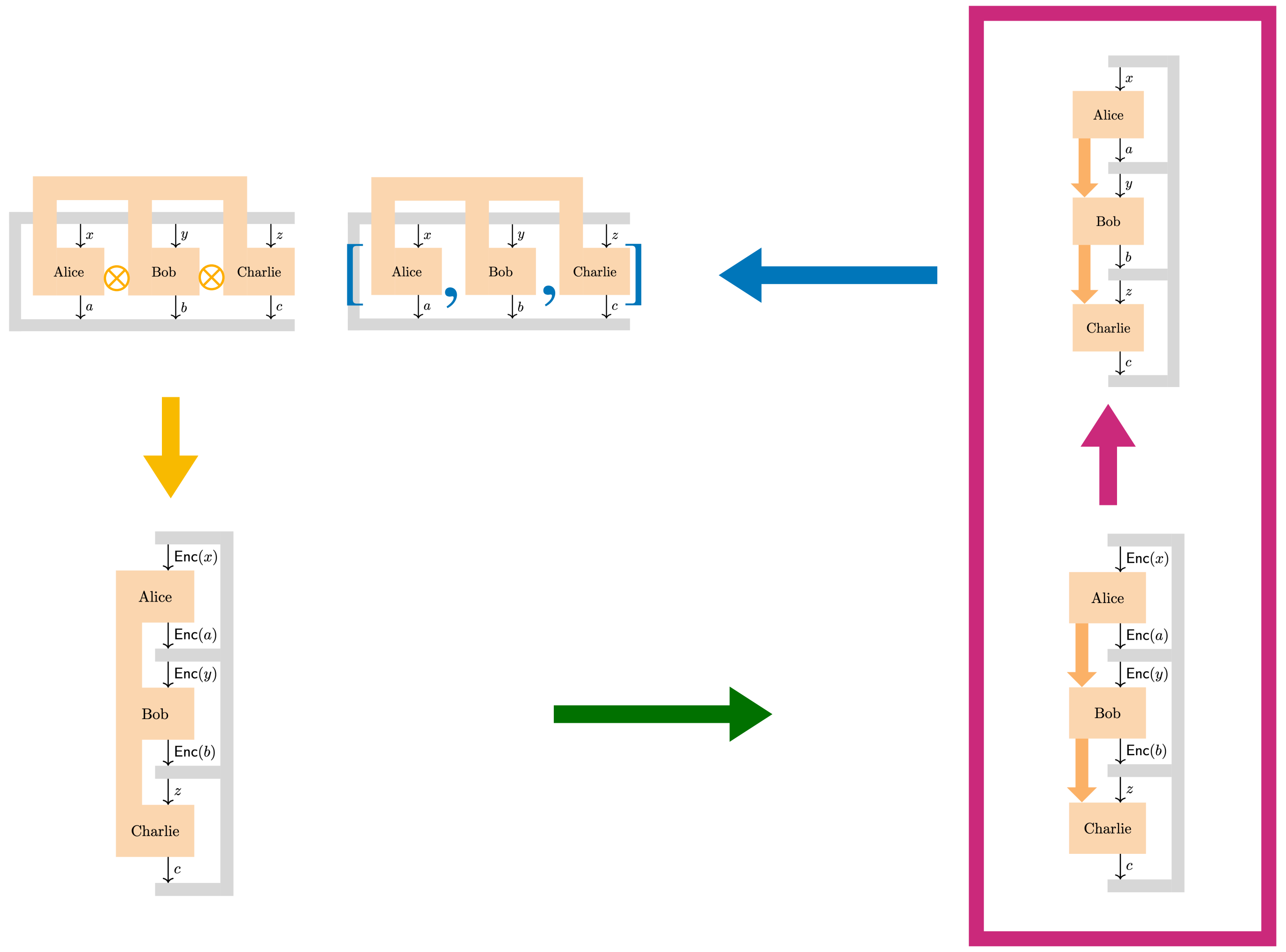
$$\begin{tikzcd} \mathscr{A}_B \arrow[d, "\theta^\lambda"'] \arrow[dr, "\phi^\lambda_{a|x}"] & \\ \mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "\mathrm{Tr}(\sigma^\lambda_{a|x}\,\cdot)"'] & \mathbb{C} \end{tikzcd} \qquad \xrightarrow{\lambda\to\infty} \qquad \phi_{a|x}$$

Universal C* algebras of PVMs

Generated by $e_{b|y} \in \mathscr{A}_B$ s.t.

1. $e_{b|y} = e^*_{b|y}$
2. $0 \leq e_{b|y} \leq 1$
3. $\sum_b e_{b|y} = 1$
4. $e_{b|y} e_{b'|y} = \delta_{b,b'} e_{b|y}$

3. The asymptotic limit



Universal C* algebras of sequential PVMs

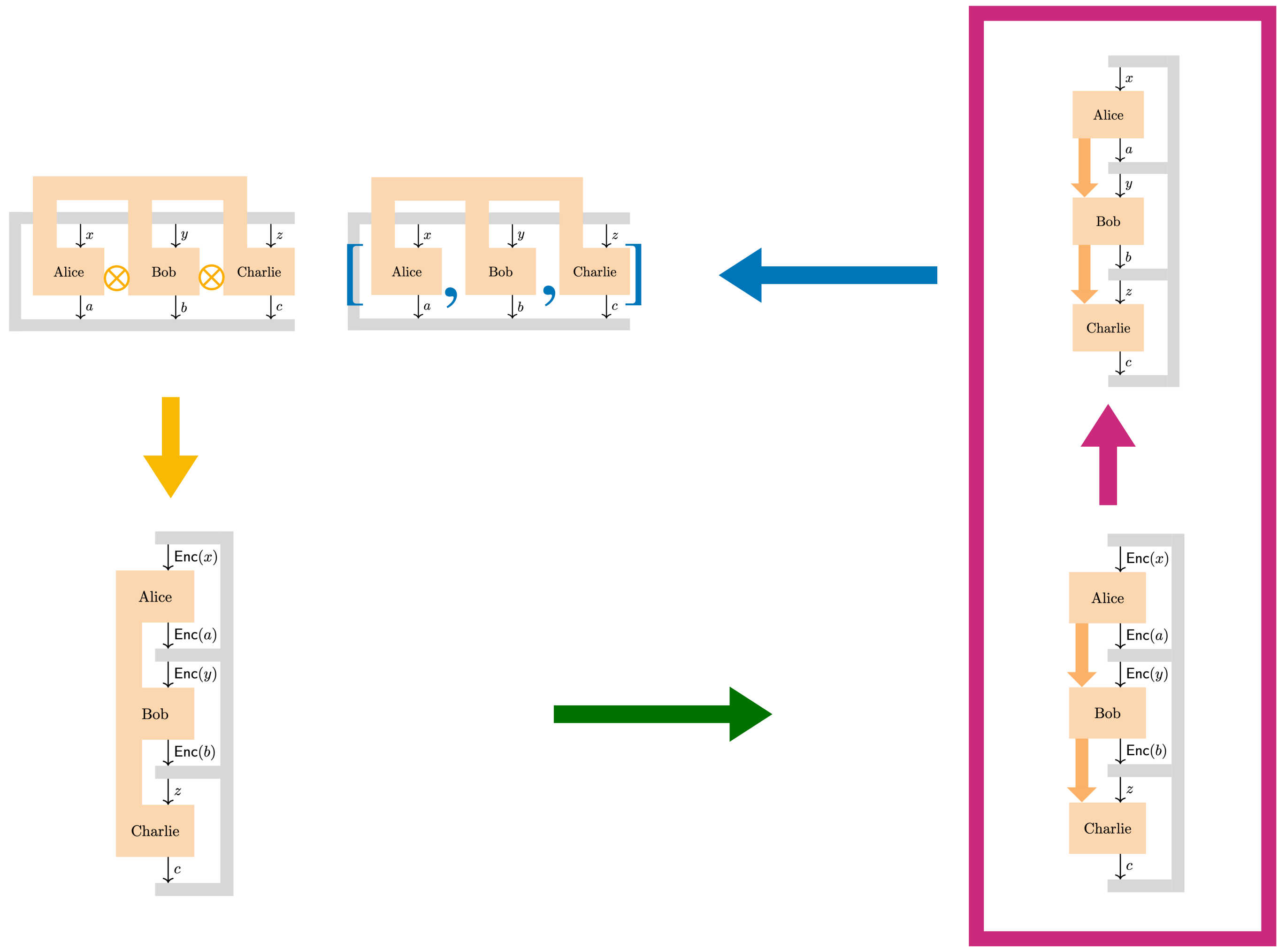
3 players

$$\mathsf{B}(\mathcal{H}^\lambda) \xrightarrow[B_{b|y}^{\lambda,*}]{} \mathsf{B}(\mathcal{H}^\lambda) \xrightarrow[\mathsf{Tr}(\sigma_{a|x}^{\lambda}\,\cdot)]{} \mathbb{C}$$

3. The asymptotic limit

Universal C* algebras of sequential PVMs

3 players

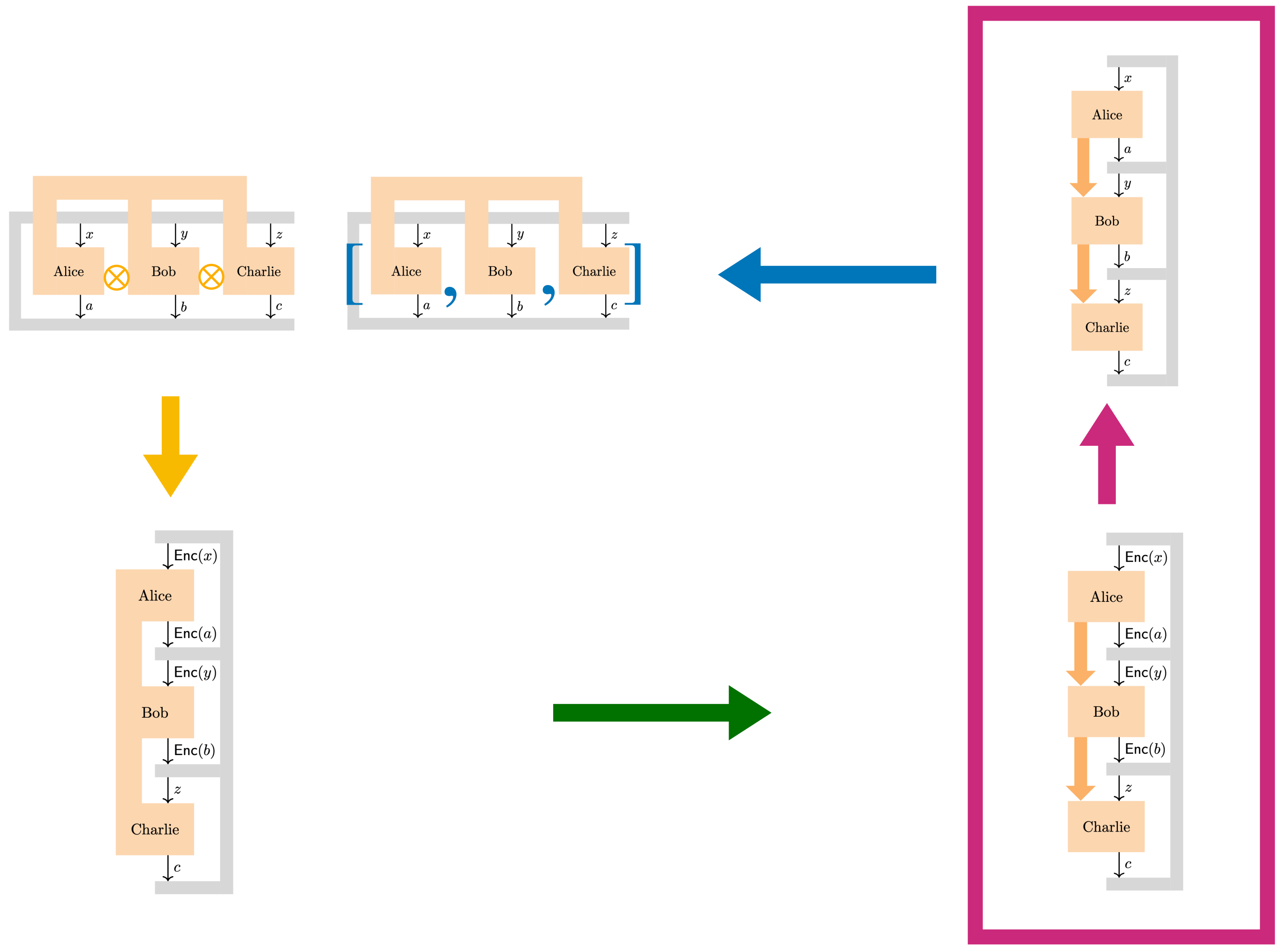


$$\begin{tikzcd}
\mathscr{A}_C \arrow[d, "\theta_C^\lambda"'] & & \\
\mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "B_{b|y}^{\lambda,*}"'] & \mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "\mathrm{Tr}(\sigma_{a|x}^\lambda \cdot)"'] & \mathbb{C}
\end{tikzcd}$$

3. The asymptotic limit

Universal C* algebras of sequential PVMs

3 players



$$\begin{tikzcd} \mathscr{A}_C \arrow[d, "\theta_C^\lambda"'] & \mathscr{A}_{B\to C} \arrow[d, "\theta_{BC}^\lambda"'] \arrow[dr, "\phi^\lambda_{a|x}"] & \\ \mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "B^{\lambda,*}_{b|y}"'] & \mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "\mathrm{Tr}(\sigma^\lambda_{a|x}\cdot)"'] & \mathbb{C} \end{tikzcd}$$

Universal C* algebras of sequential PVMs

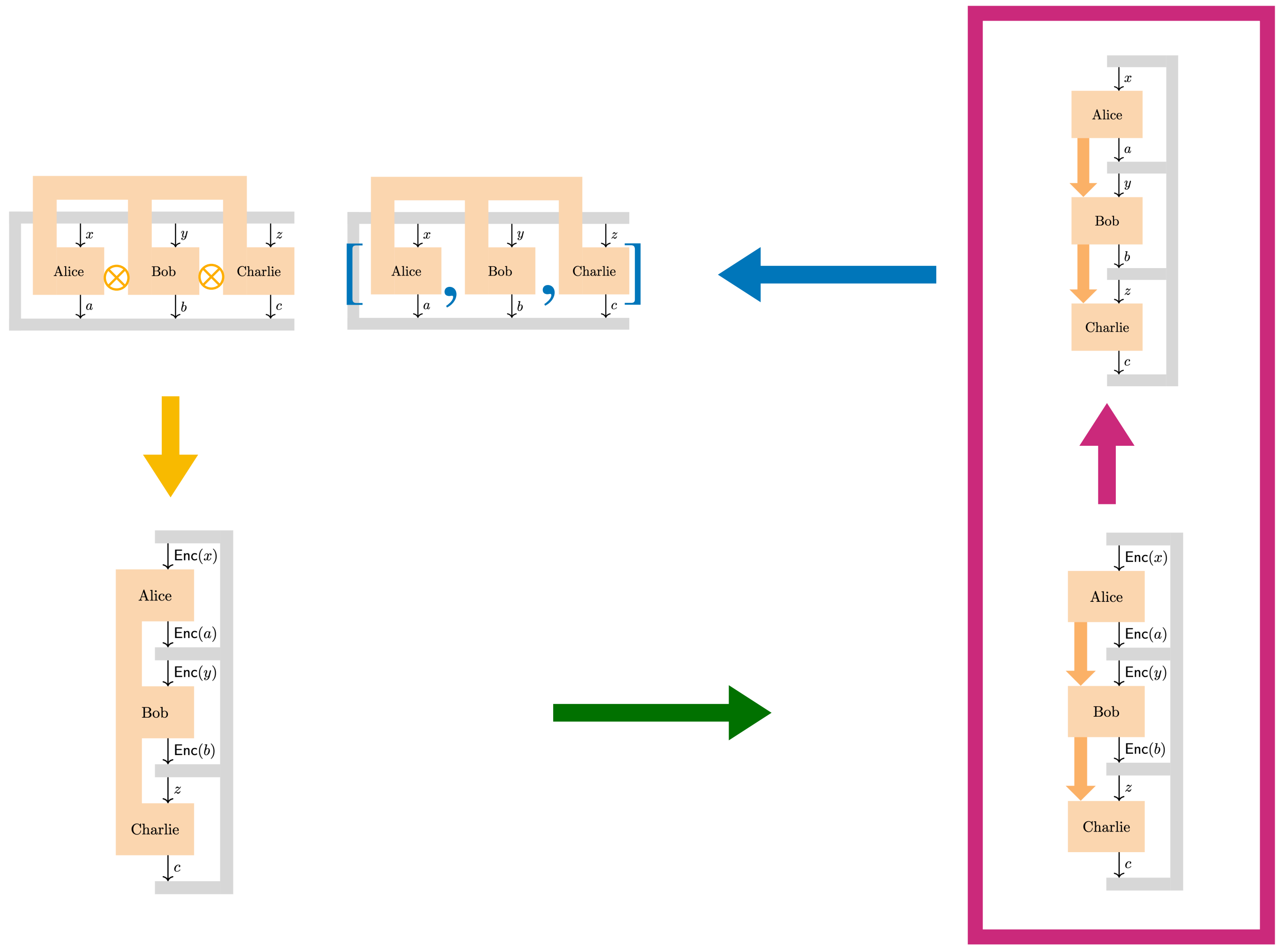
Generated by $f_{bc|yz} \in \mathscr{A}_{B\to C}$ s.t.

1. $f_{bc|yz} = f^*_{bc|yz}$
2. $0 \leq f_{bc|yz} \leq 1$
3. $\sum_{b,c} f_{bc|yz} = 1$
4. $f_{bc|yz} f_{b'c'|yz} = \delta_{b,b'}\delta_{c,c'} f_{bc|yz}$
5. $\sum_{c} f_{bc|yz} = \sum_{c} f_{bc|yz'}$

3. The asymptotic limit

Universal C* algebras of sequential PVMs

3 players



$$
\begin{tikzcd}
\mathscr{A}_C \arrow[d, "\theta_C^\lambda"'] & & \mathscr{A}_{B\to C} \arrow[d, "\theta_{BC}^\lambda"'] \arrow[drr, "\phi_{a|x}^\lambda"] & & \\
\mathrm{B}(\mathcal{H}^\lambda) \arrow[rr, "B_{b|y}^{\lambda,*}"'] & & \mathrm{B}(\mathcal{H}^\lambda) \arrow[rr, "\mathrm{Tr}(\sigma_{a|x}^\lambda\,\cdot)"'] & & \mathbb{C}
\end{tikzcd}
$$

Universal C* algebras of sequential PVMs

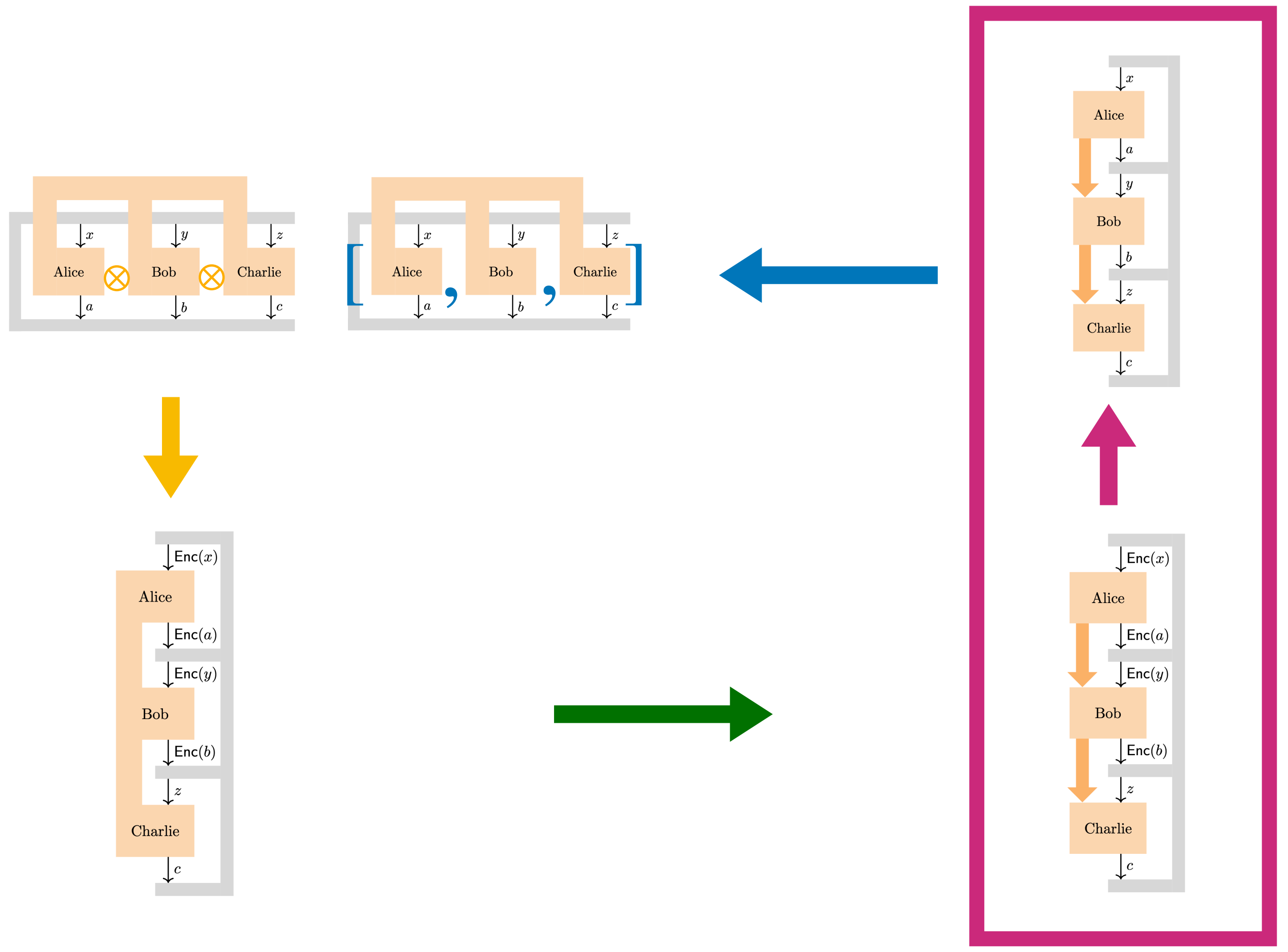
Generated by $f_{bc|yz} \in \mathscr{A}_{B\to C}$ s.t.

1. $f_{bc|yz} = f_{bc|yz}^*$
2. $0 \leq f_{bc|yz} \leq 1$
3. $\sum_{b,c} f_{bc|yz} = 1$
4. $f_{bc|yz} f_{b'c'|yz} = \delta_{b,b'}\delta_{c,c'} f_{bc|yz}$
5. $\sum_{c} f_{bc|yz} = \sum_{c} f_{bc|yz'}$

3. The asymptotic limit

Universal C* algebras of sequential PVMs

3 players



$$\begin{tikzcd}
\mathscr{A}_C \arrow[r, "T_{b|y}"] \arrow[d, "\theta_C^\lambda"'] & \mathscr{A}_{B\to C} \arrow[d, "\theta^\lambda_{BC}"'] \arrow[dr, "\phi^\lambda_{a|x}"] & \\
\mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "B^{\lambda,*}_{b|y}"'] & \mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "\mathrm{Tr}(\sigma^\lambda_{a|x}\cdot)"'] & \mathbb{C}
\end{tikzcd}$$

Universal C* algebras of sequential PVMs

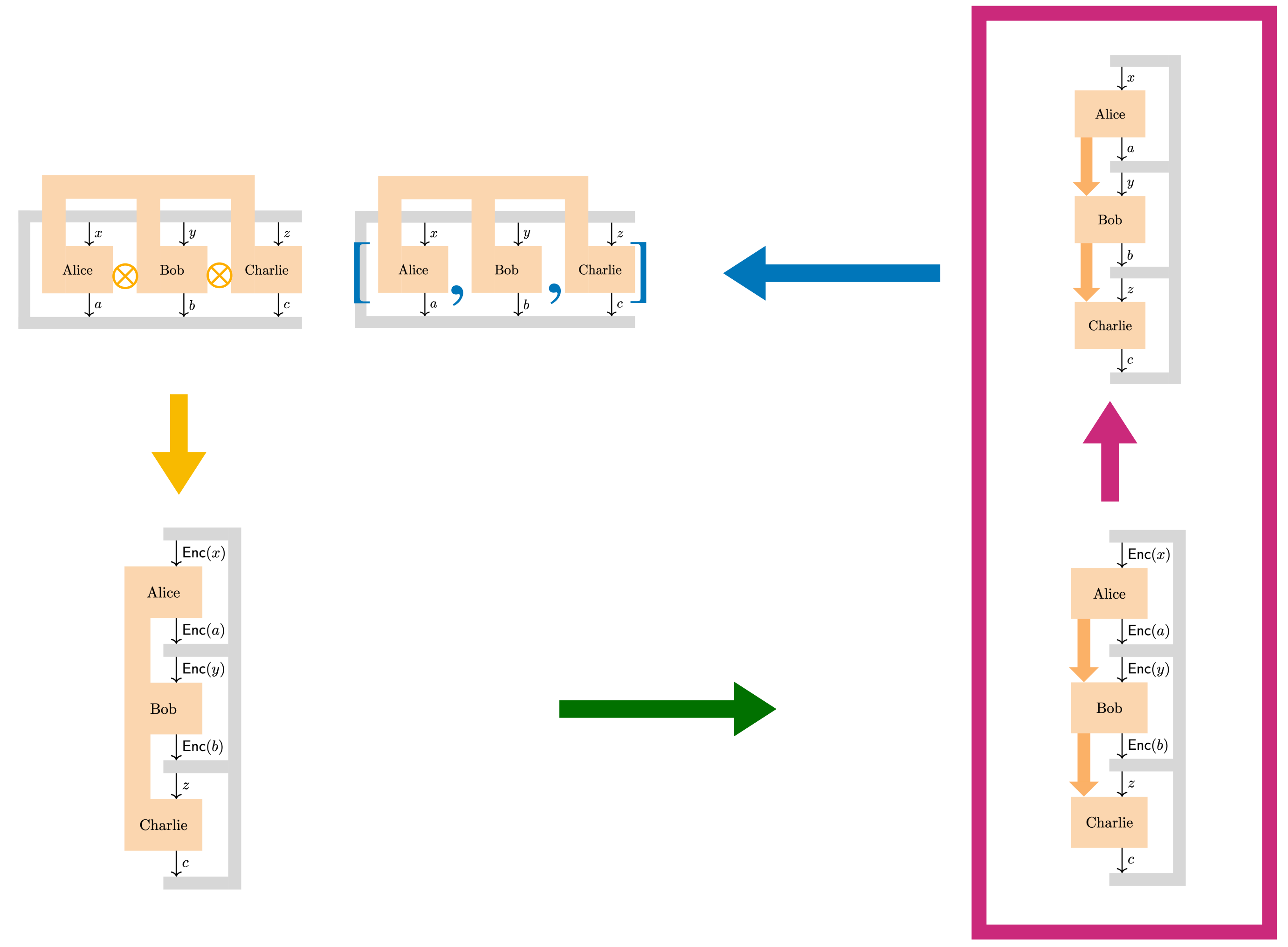
Generated by $f_{bc|yz} \in \mathscr{A}_{B\to C}$ s.t.

1. $f_{bc|yz} = f^*_{bc|yz}$
2. $0 \leq f_{bc|yz} \leq 1$
3. $\sum_{b,c} f_{bc|yz} = 1$
4. $f_{bc|yz} f_{b'c'|yz} = \delta_{b,b'}\delta_{c,c'} f_{bc|yz}$
5. $\sum_c f_{bc|yz} = \sum_c f_{bc|yz'}$

3. The asymptotic limit

Universal C* algebras of sequential PVMs

3 players



$$
\begin{tikzcd}
\mathscr{A}_C \arrow[r, "T_{b|y}"] \arrow[d, "\theta_C^\lambda"'] & \mathscr{A}_{B\to C} \arrow[d, "\theta_{BC}^\lambda"'] \arrow[dr, "\phi^\lambda_{a|x}"] & \\
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\end{tikzcd}
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Universal C* algebras of sequential PVMs

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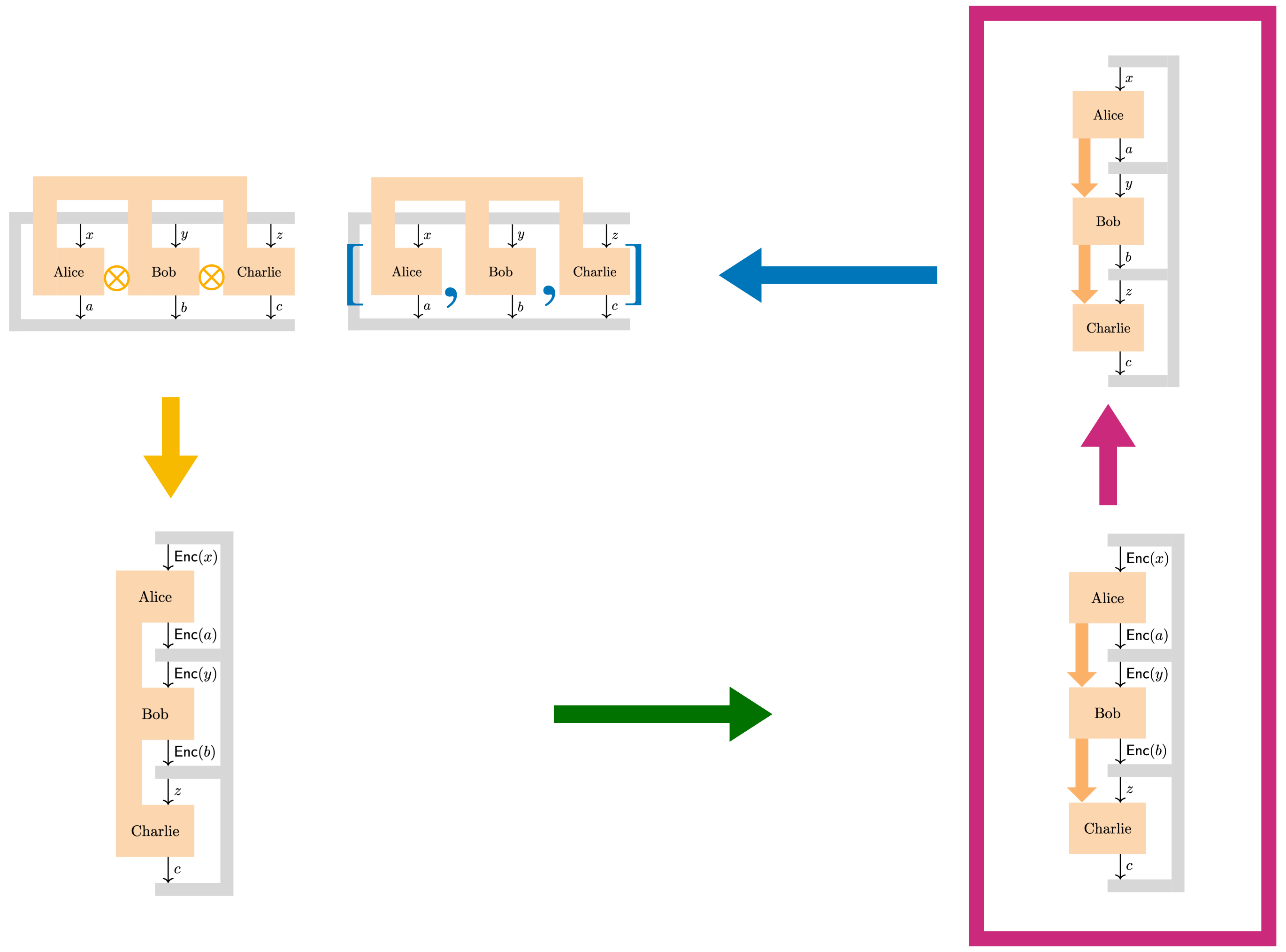
5. $\sum_c f_{bc|yz} = \sum_c f_{bc|yz'}$

3. The asymptotic limit

Universal C* algebras of sequential PVMs

3 players



$$\begin{tikzcd}
\mathscr{A}_C \arrow[r, "T_{b|y}"] \arrow[d, "\theta_C^\lambda"'] & \mathscr{A}_{B\to C} \arrow[d, "\theta_{BC}^\lambda"'] \arrow[dr, "\phi^\lambda_{a|x}"] & \\
\mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "B^{\lambda,*}_{b|y}"'] & \mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "\mathrm{Tr}(\sigma^\lambda_{a|x}\,\cdot)"'] & \mathbb{C}
\end{tikzcd}$$

$$\phi^\lambda_{a|x} \xrightarrow{\lambda\to\infty} \phi_{a|x}$$

Universal C* algebras of sequential PVMs

Generated by $f_{bc|yz} \in \mathscr{A}_{B\to C}$ s.t.

1. $f_{bc|yz} = f^*_{bc|yz}$
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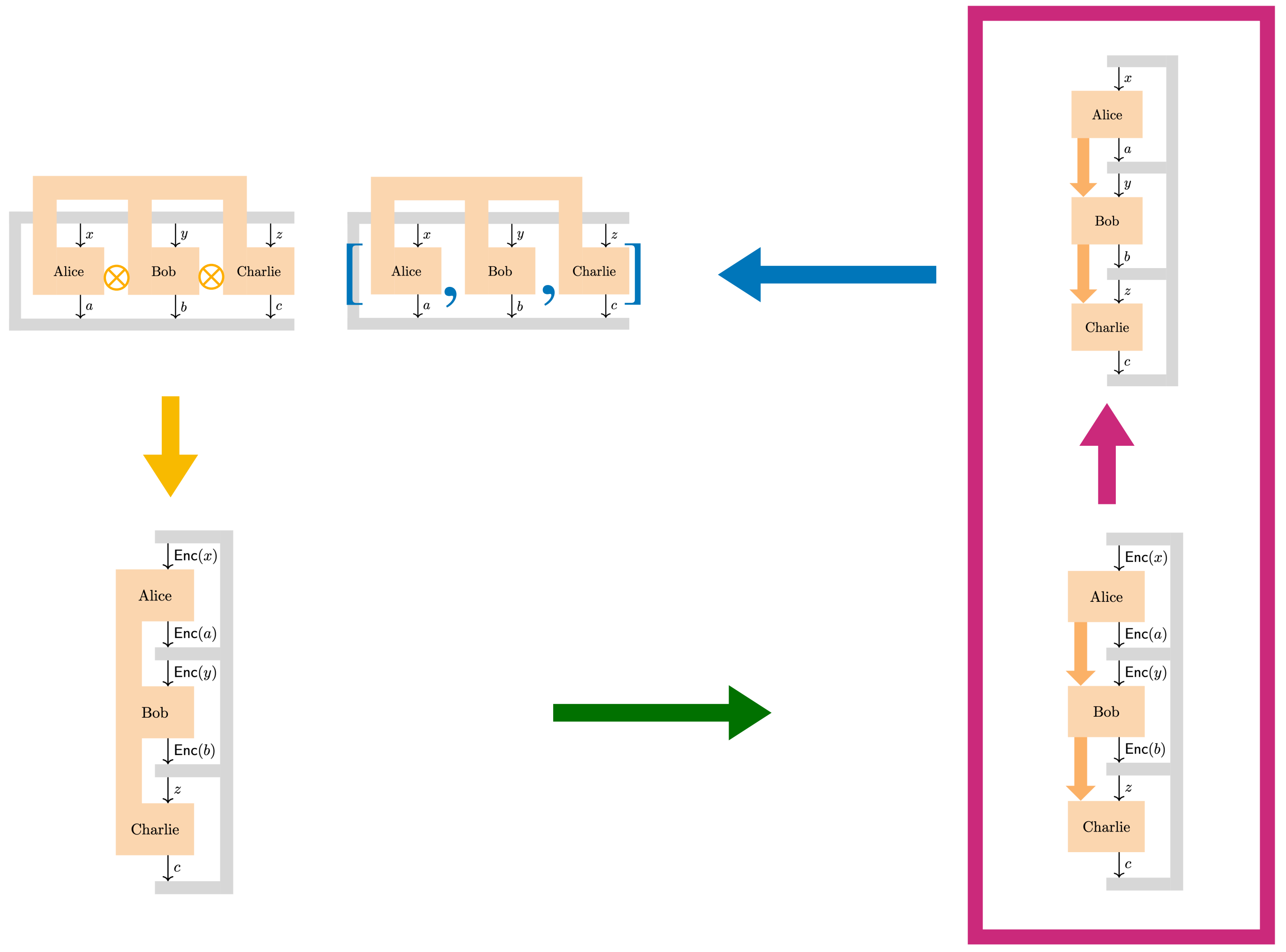
3. $\sum_{b,c} f_{bc|yz} = 1$
4. $f_{bc|yz} f_{b'c'|yz} = \delta_{b,b'}\delta_{c,c'} f_{bc|yz}$
5. $\sum_c f_{bc|yz} = \sum_c f_{bc|yz'}$

3. The asymptotic limit

Universal C* algebras of sequential PVMs

3 players



$$
\begin{tikzcd}
\mathscr{A}_C \arrow[r, "T_{b|y}"] \arrow[d, "\theta^\lambda_C"'] & \mathscr{A}_{B\to C} \arrow[d, "\theta^\lambda_{BC}"'] \arrow[dr, "\phi^\lambda_{a|x}"] & \\
\mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "B^{\lambda,*}_{b|y}"'] & \mathrm{B}(\mathcal{H}^\lambda) \arrow[r, "\mathrm{Tr}(\sigma^\lambda_{a|x}\cdot)"'] & \mathbb{C}
\end{tikzcd}
$$

$$\phi^\lambda_{a|x} \xrightarrow{\lambda\to\infty} \phi_{a|x}$$

Universal C* algebras of sequential PVMs

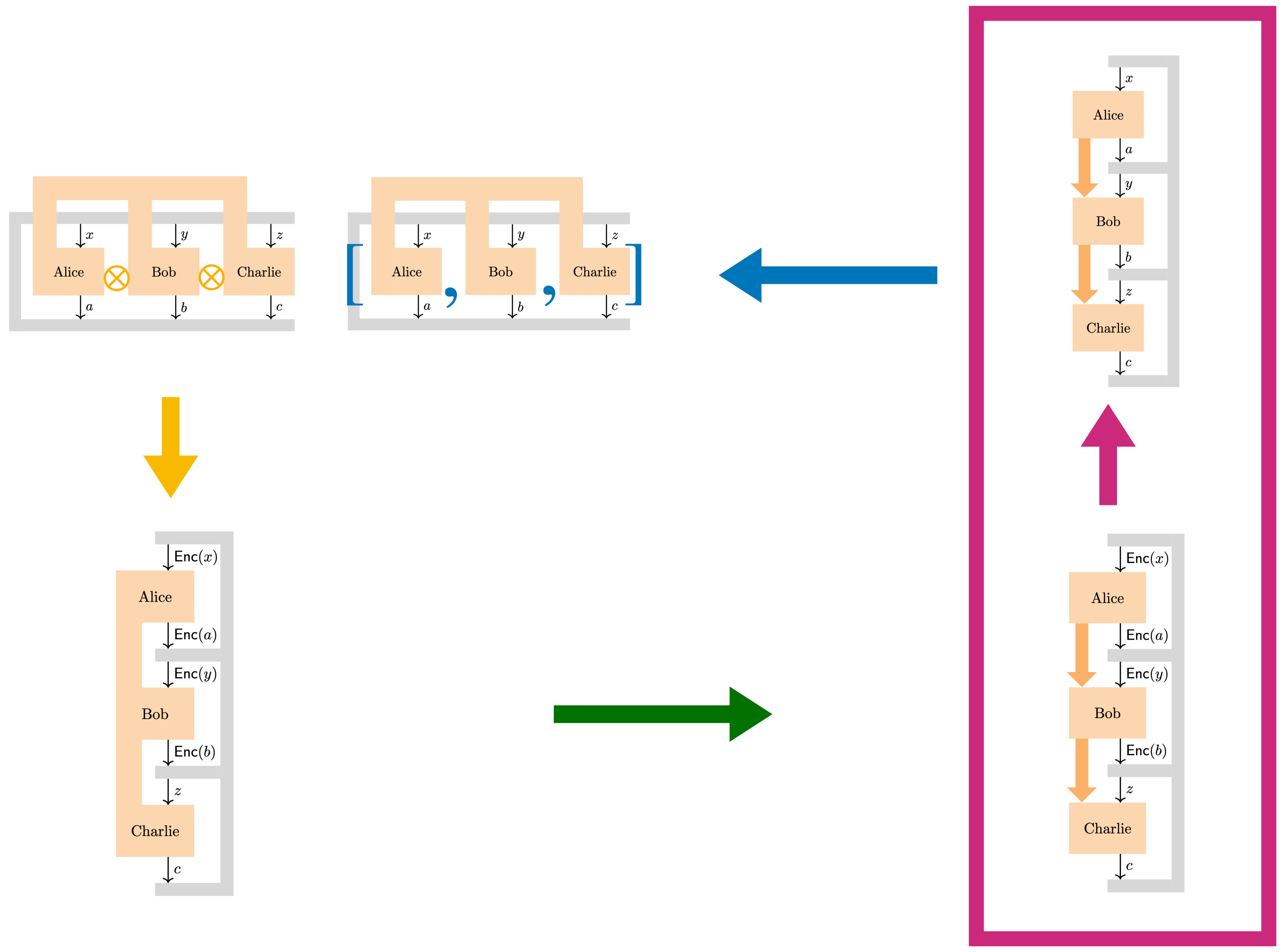
Generated by $f_{bc|yz} \in \mathscr{A}_{B\to C}$ s.t.

1. $f_{bc|yz} = f^*_{bc|yz}$
2. $0 \le f_{bc|yz} \le 1$
3. $\sum_{b,c} f_{bc|yz} = 1$
4. $f_{bc|yz} f_{b'c'|yz} = \delta_{b,b'}\delta_{c,c'} f_{bc|yz}$
5. $\sum_c f_{bc|yz} = \sum_c f_{bc|yz'}$

3. The asymptotic limit

Universal C* algebras of sequential PVMs

3 players



$$\begin{tikzcd}
\mathscr{A}_C \arrow[r, "T_{b|y}"] \arrow[d, "\theta_C^\lambda"'] & \mathscr{A}_{B\to C} \arrow[d, "\theta_{BC}^\lambda"] \arrow[dr, "\phi^\lambda_{a|x}"] & \\
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\end{tikzcd}$$

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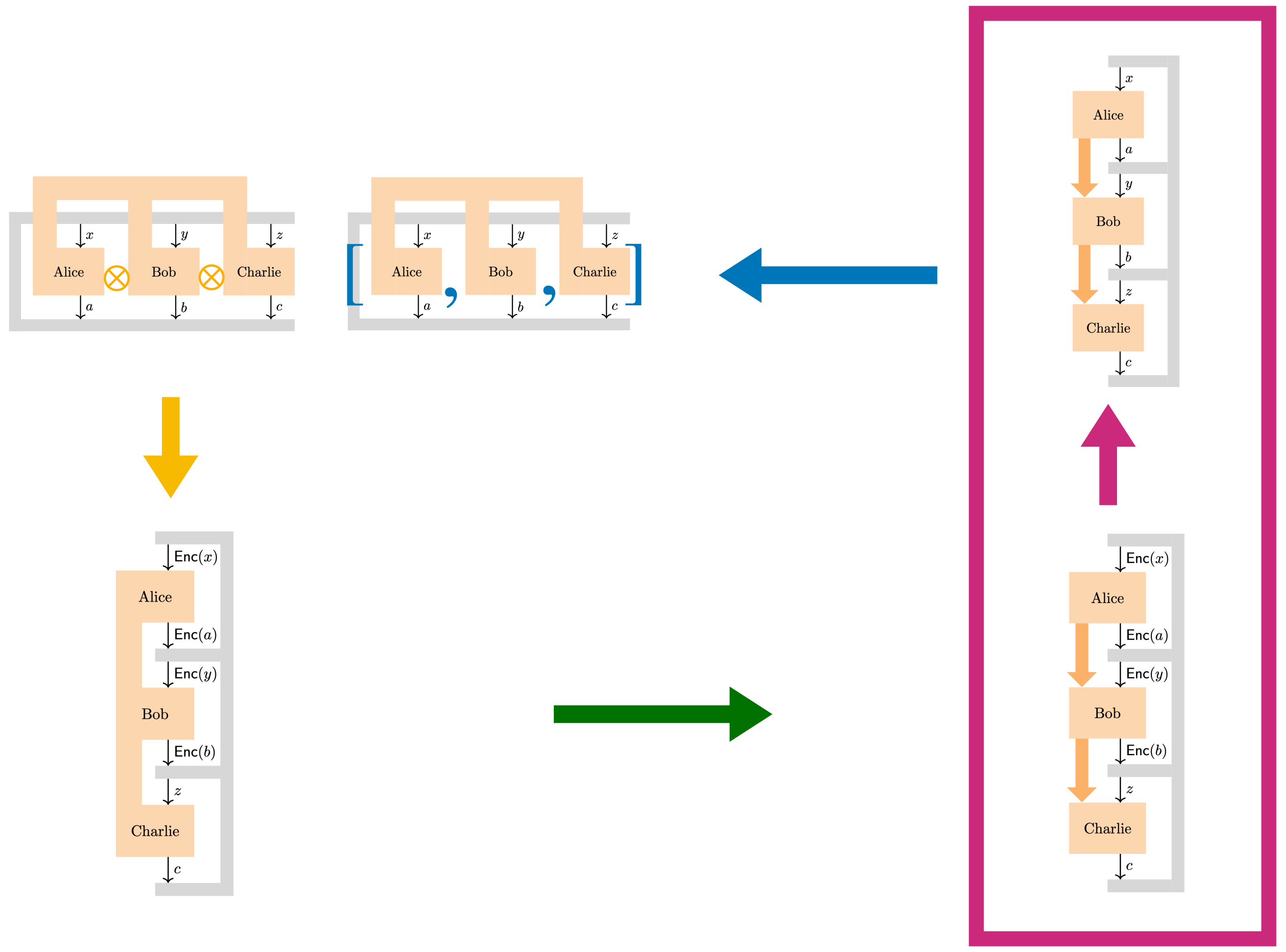
Universal C* algebras of sequential PVMs



3 players

$$\phi^\lambda_{a|x} \circ T_y \approx_\lambda \phi^\lambda_{a|x} \circ T_{y'}$$

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Universal C* algebras of sequential PVMs

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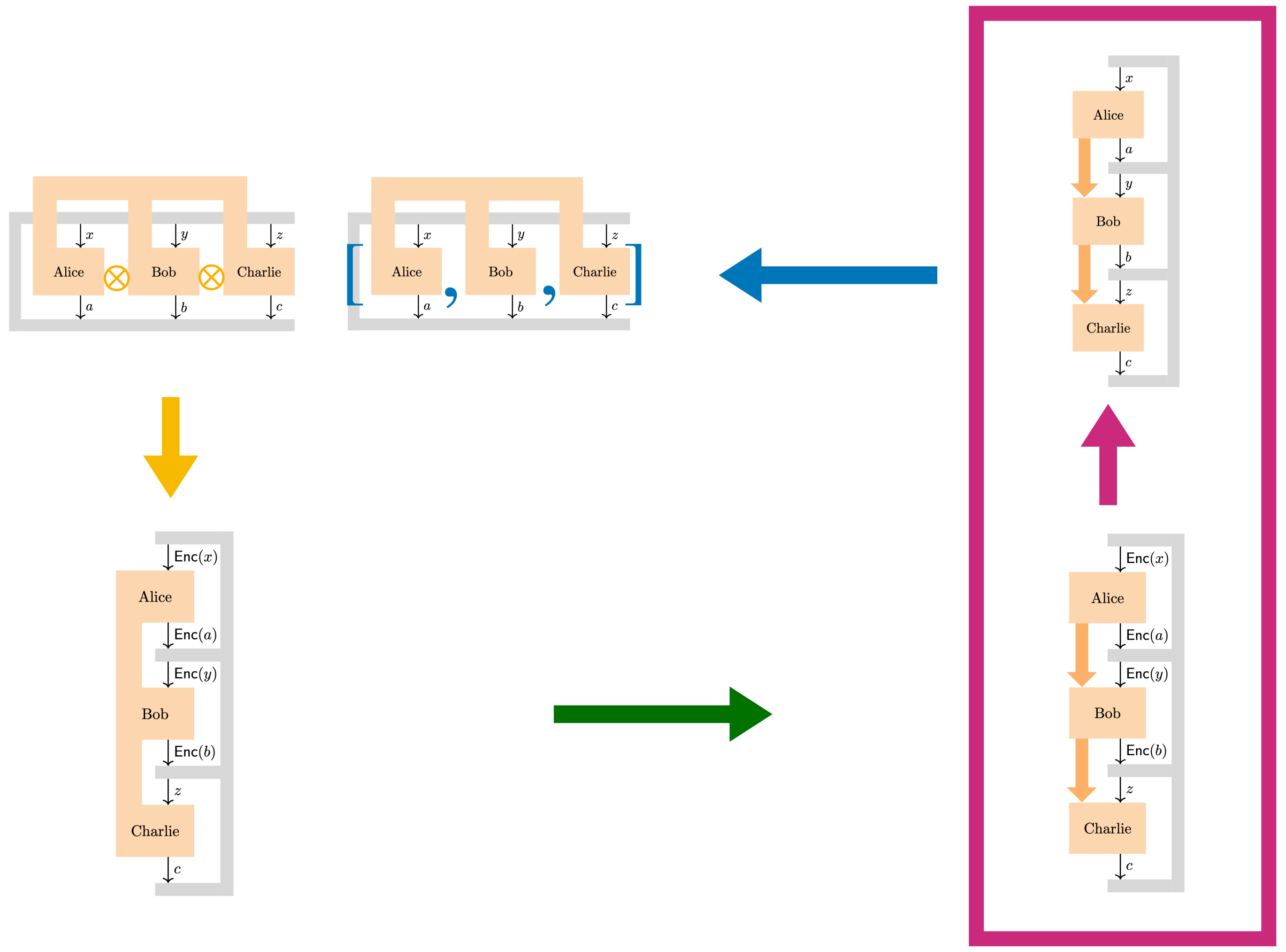
Asymptotic constraints



Approximate constraints from IND-CPA

$\lambda \to \infty$

Exact constraints

$$\phi_x^\lambda \approx_\lambda \phi_{x'}^\lambda$$

$$\phi_x = \phi_{x'}$$

$$\phi_{a|x}^\lambda \circ T_y \approx_\lambda \phi_{a|x}^\lambda \circ T_{y'}$$

$$\phi_{a|x} \circ T_y = \phi_{a|x} \circ T_{y'}$$

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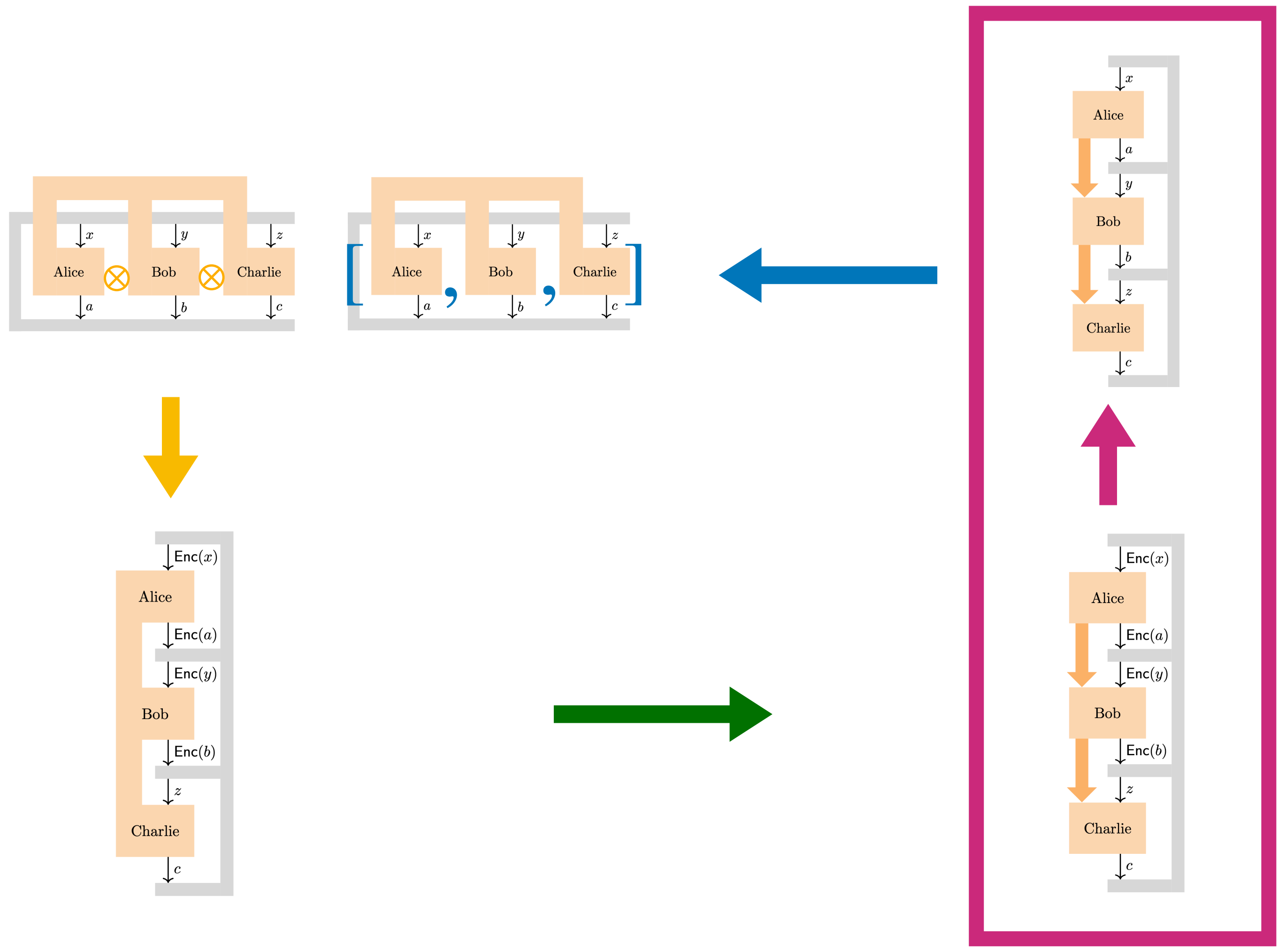
Asymptotic constraints



Approximate constraints from IND-CPA

$$\phi^{\lambda}_{x} \approx_{\lambda} \phi^{\lambda}_{x'}$$

$$\phi^{\lambda}_{a|x} \circ T_y \approx_{\lambda} \phi^{\lambda}_{a|x} \circ T_{y'}$$

$\lambda \to \infty$

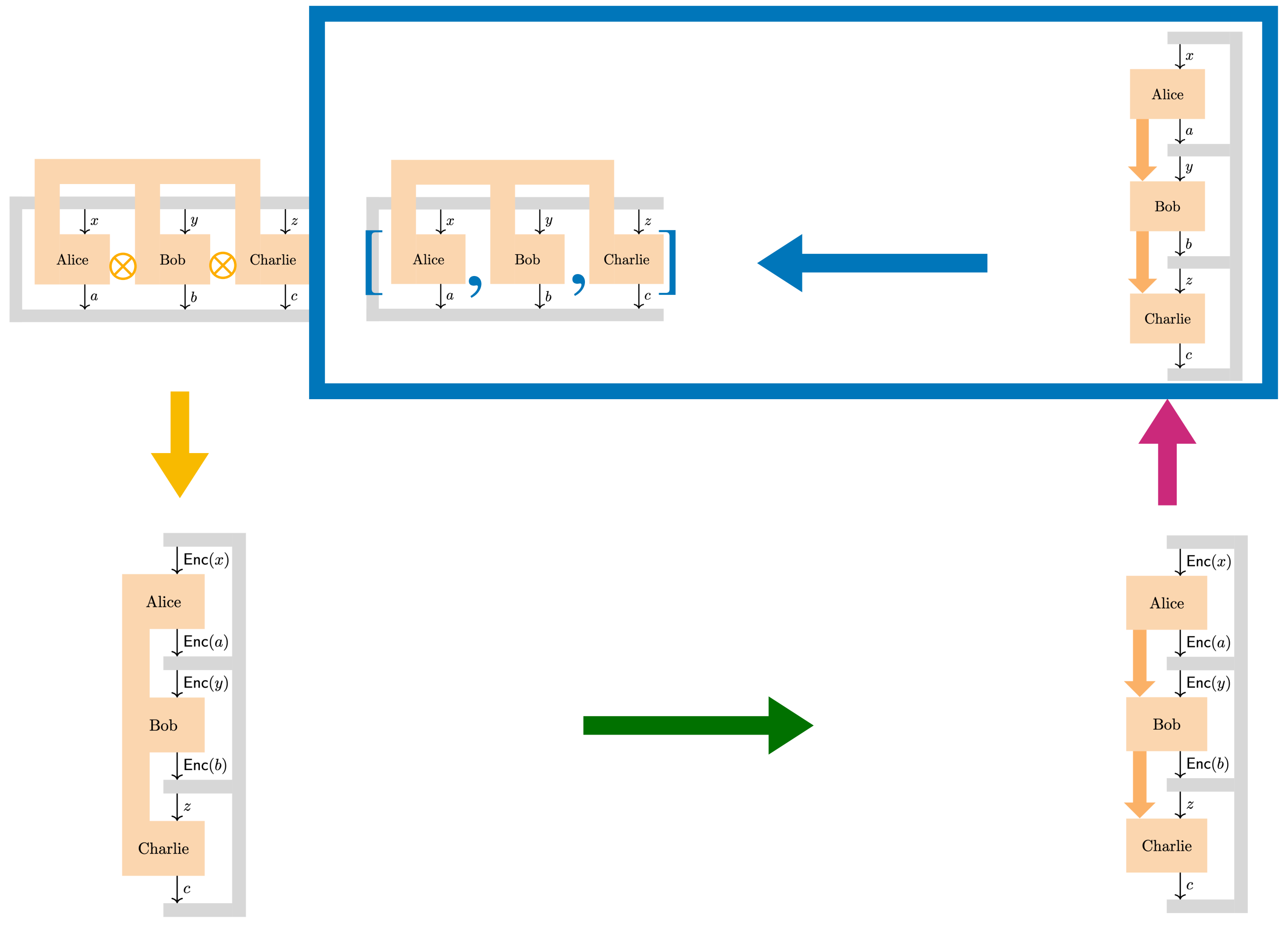
Exact constraints

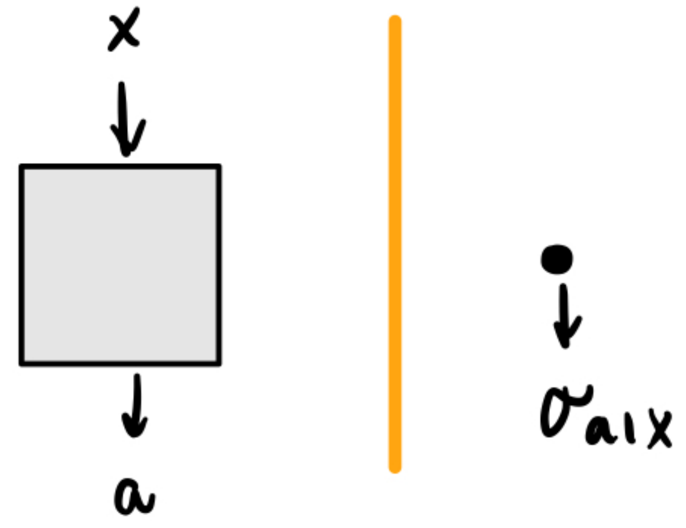
$$\phi_x = \phi_{x'}$$

$$T_y = T_{y'}$$

4. From sequential to non-local

Post-quantum steering arXiv: 1505.01430





$$\sum_a \sigma_{a|x} = \sigma \quad \forall x$$

$$\Downarrow$$

$$\sigma_{a|x} = \mathrm{tr}_K\left(\left[A^K_{a|x} \otimes \mathbb{1}\right]\rho\right)$$

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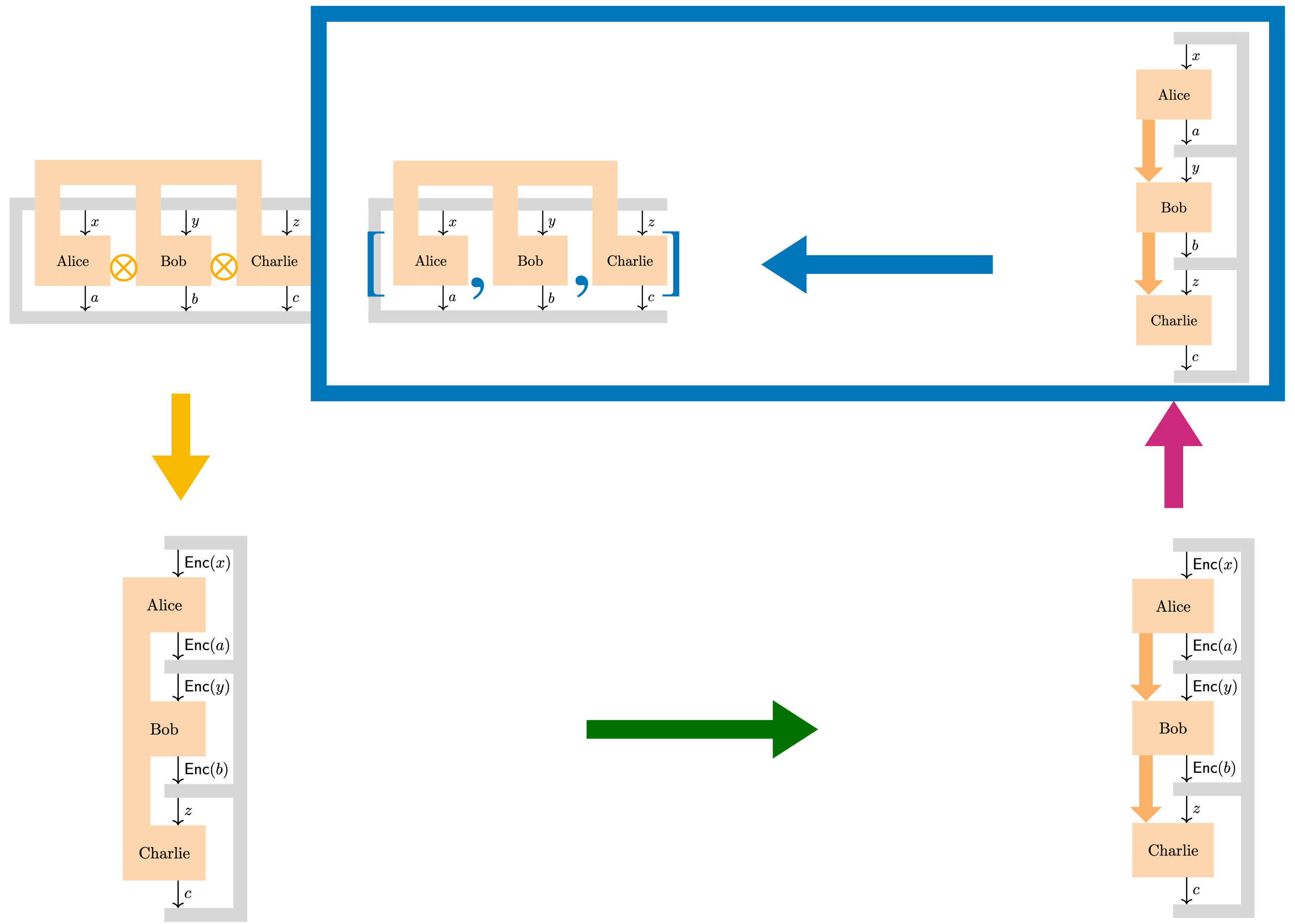

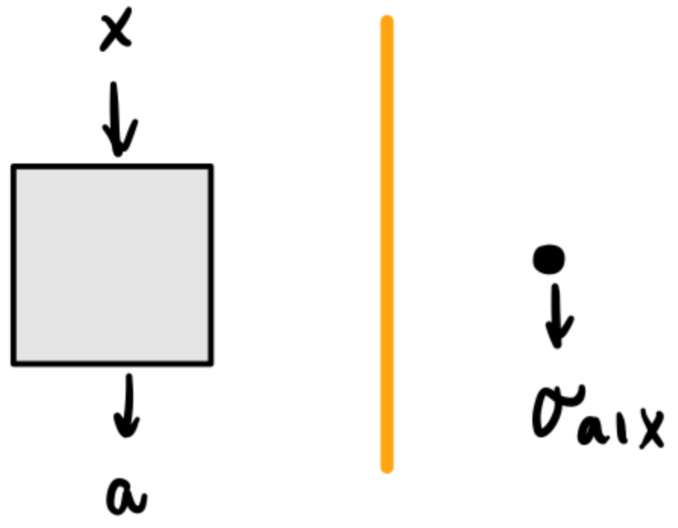

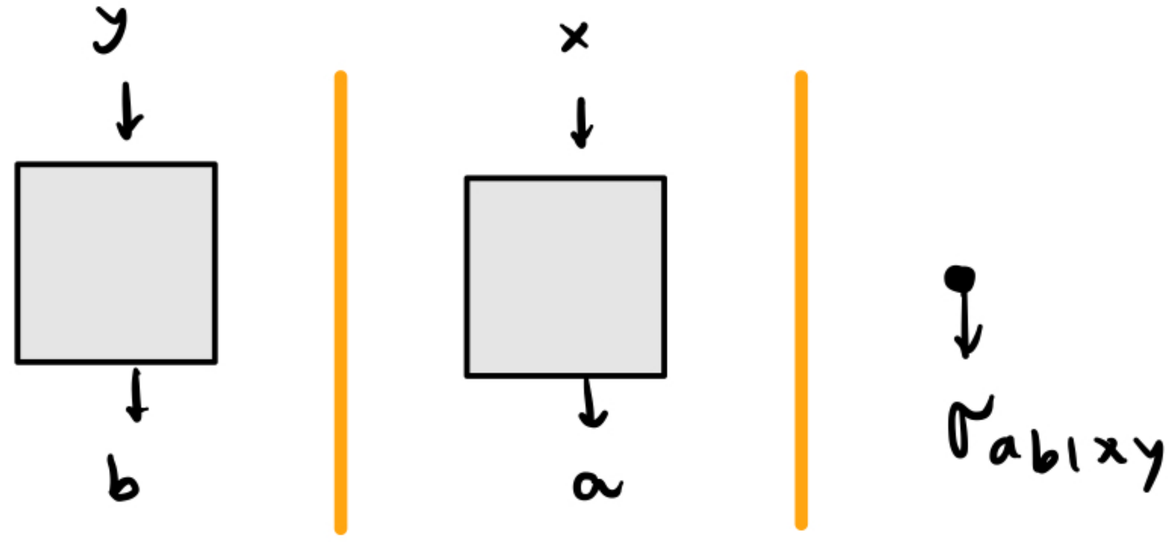

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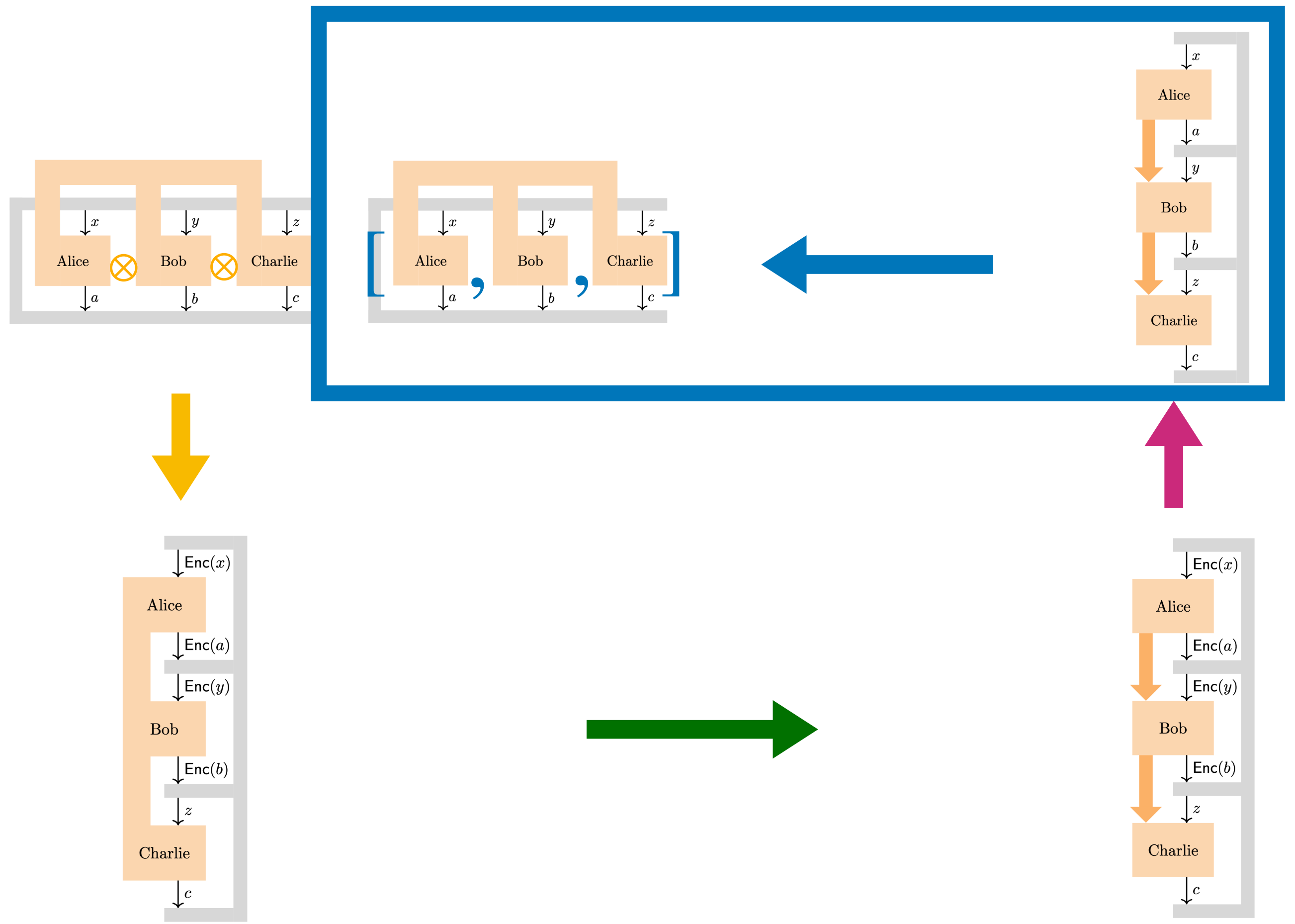

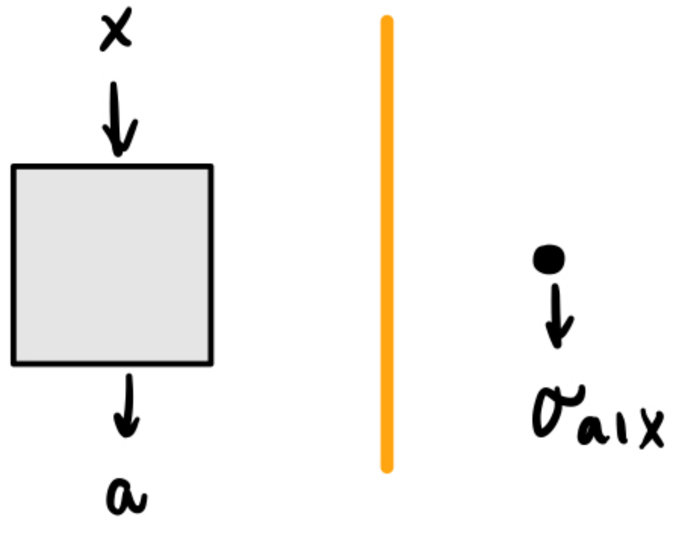

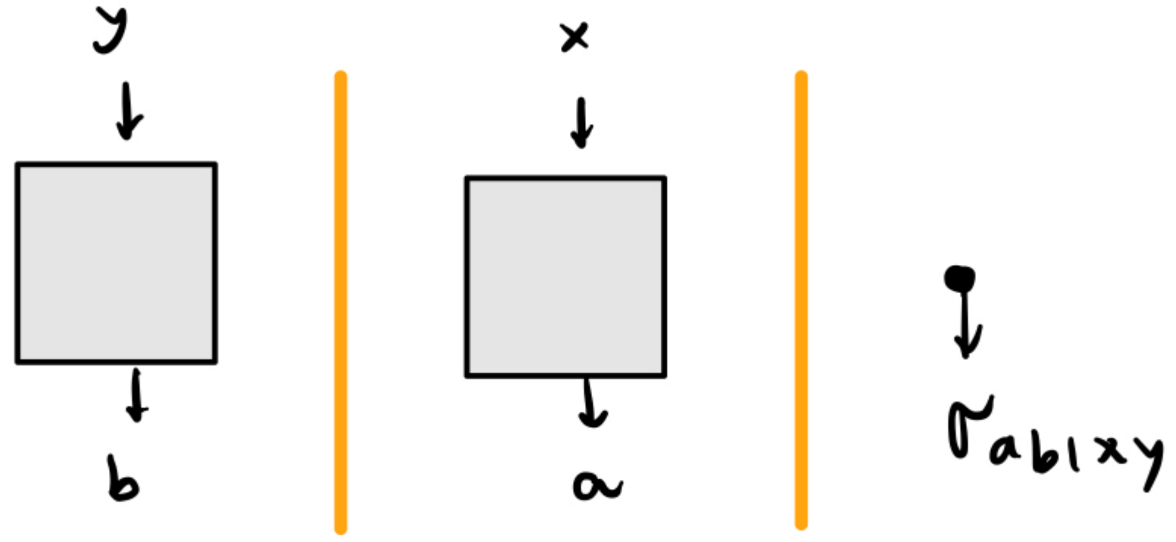

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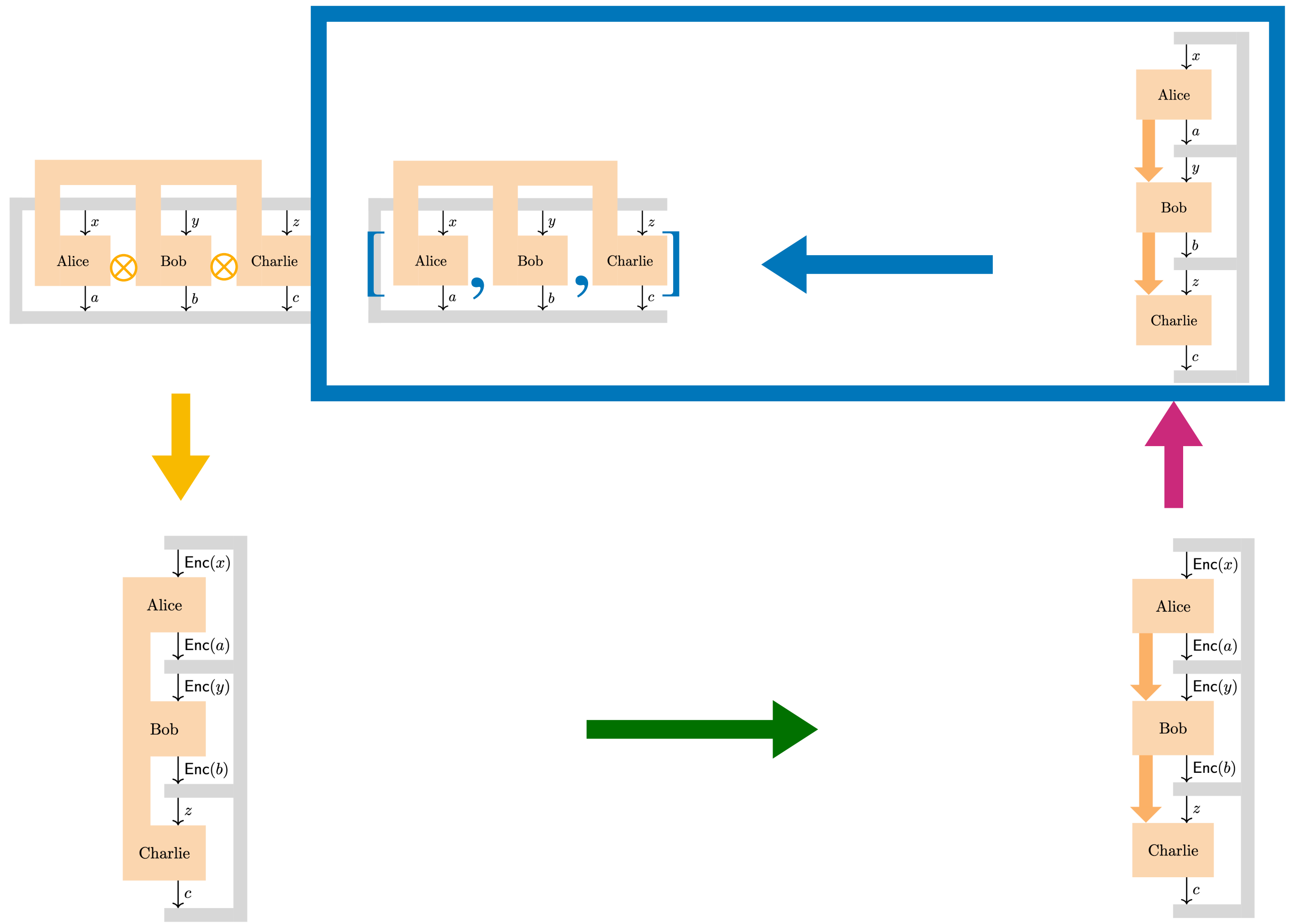

Radon-Nikodym (RN) theorem

2 players Only preparation equivalences

S-G-HJW theorem

$$\sum_a \rho_{a|x} = \rho \qquad \forall x \qquad \Leftrightarrow \qquad \rho_{a|x} = \mathrm{Tr}_k\left((A_{a|x} \otimes \mathbb{1})\sigma\right)$$

4. From sequential to non-local

Radon-Nikodym (RN) theorem

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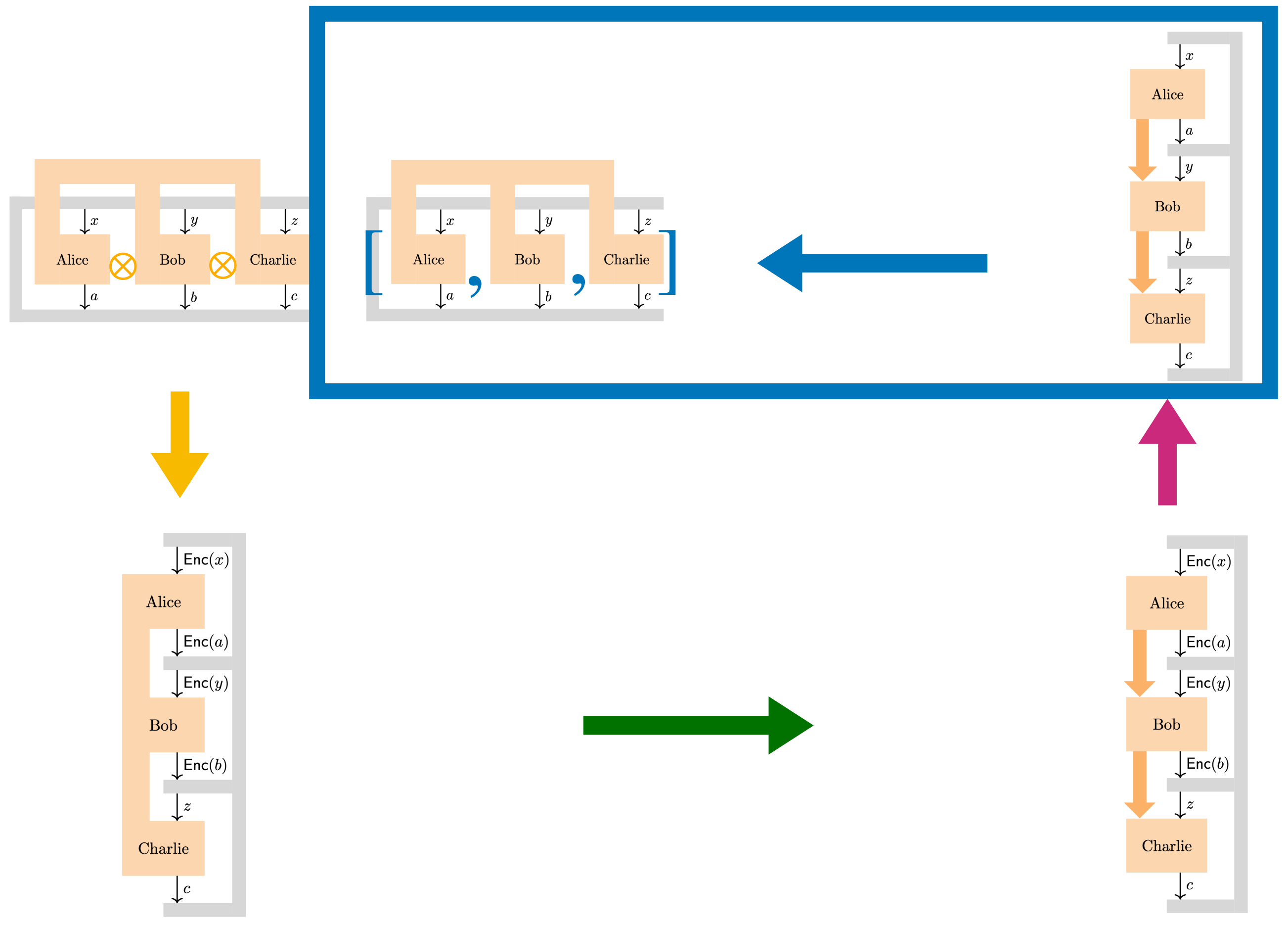

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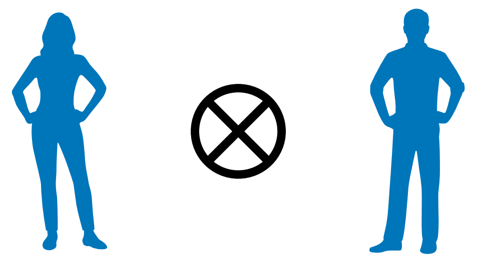
$$\begin{aligned}\mathrm{Tr}_h[B_{b|y}\rho_{a|x}] &= \mathrm{Tr}_h[B_{b|y}\mathrm{Tr}_k\left((A_{a|x} \otimes \mathbb{1})\sigma\right)] \\ &= \mathrm{Tr}_{h\otimes k}\left[(A_{a|x} \otimes B_{b|y})\sigma\right]\end{aligned}$$



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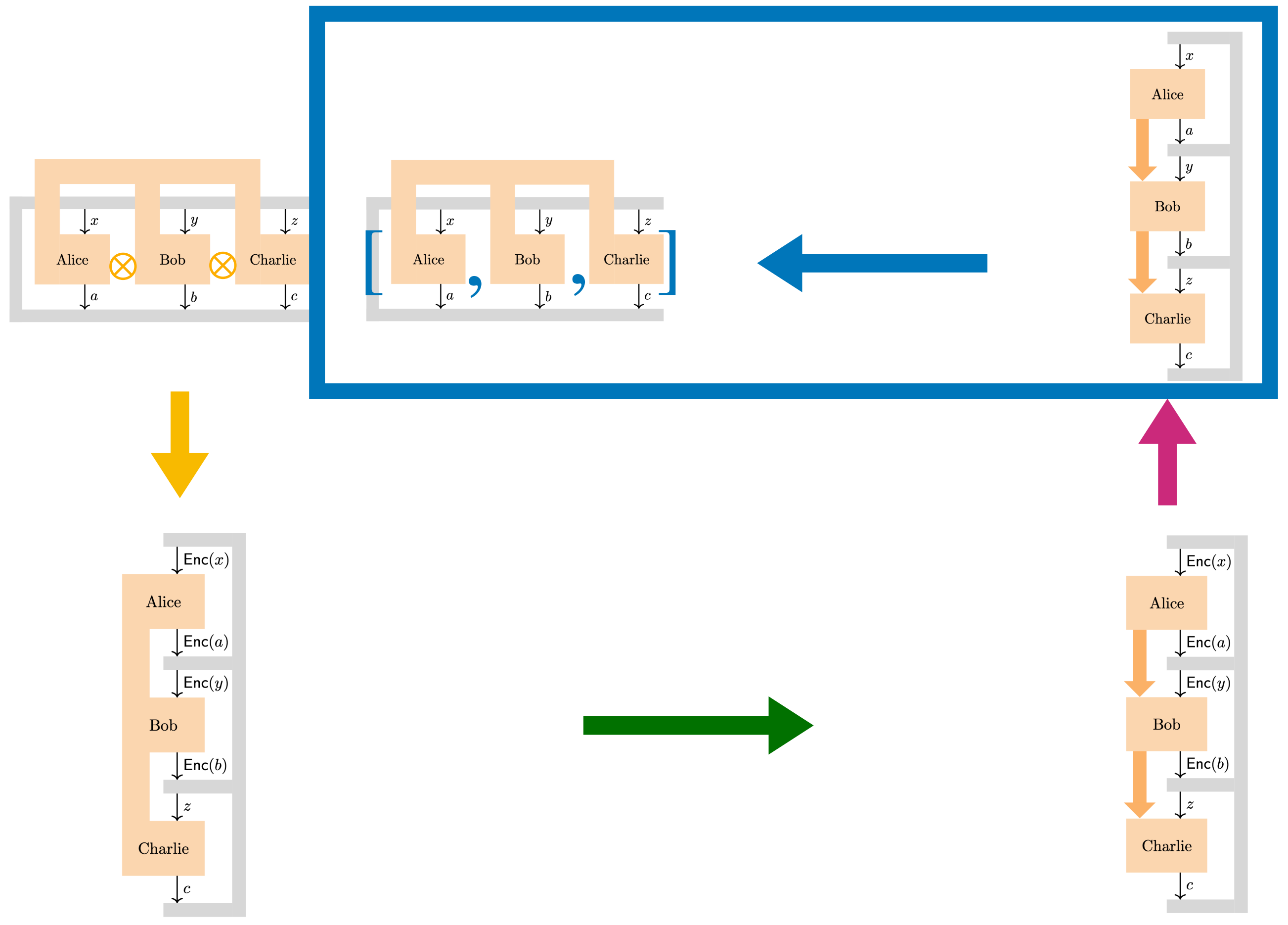

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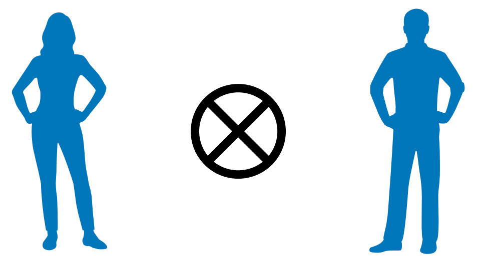


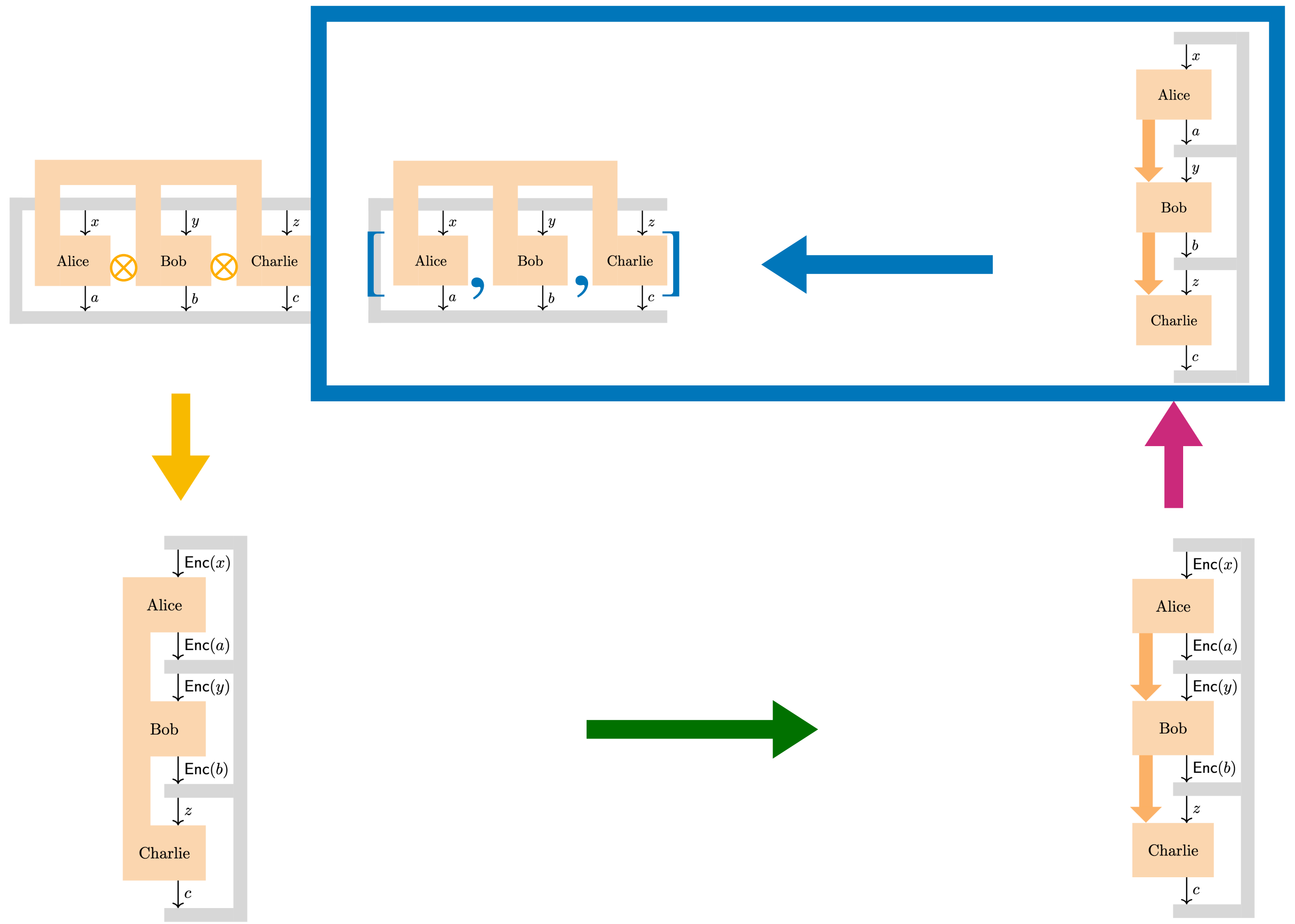
RN theorem (adapted)

$$\sum_a \phi_{a|x}(\mathfrak{a}) = \phi(\mathfrak{a}) \qquad \forall x \qquad \Leftrightarrow \qquad \begin{aligned}&\phi_{a|x}(\mathfrak{a}) = \langle \Omega_\phi | \, D_{a|x}\pi_\phi(\mathfrak{a}) \, | \Omega_\phi \rangle \\ &[D_{a|x}, \pi_\phi(\mathfrak{a})] = 0 \qquad \forall \mathfrak{a} \in \mathscr{A}\end{aligned}$$

4. From sequential to non-local

Radon-Nikodym (RN) theorem

2 players Only preparation equivalences



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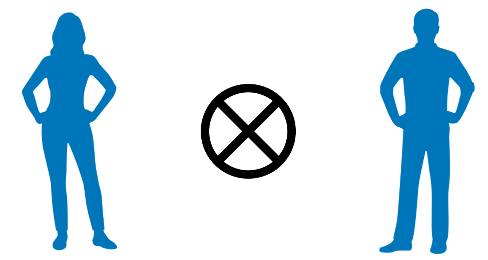

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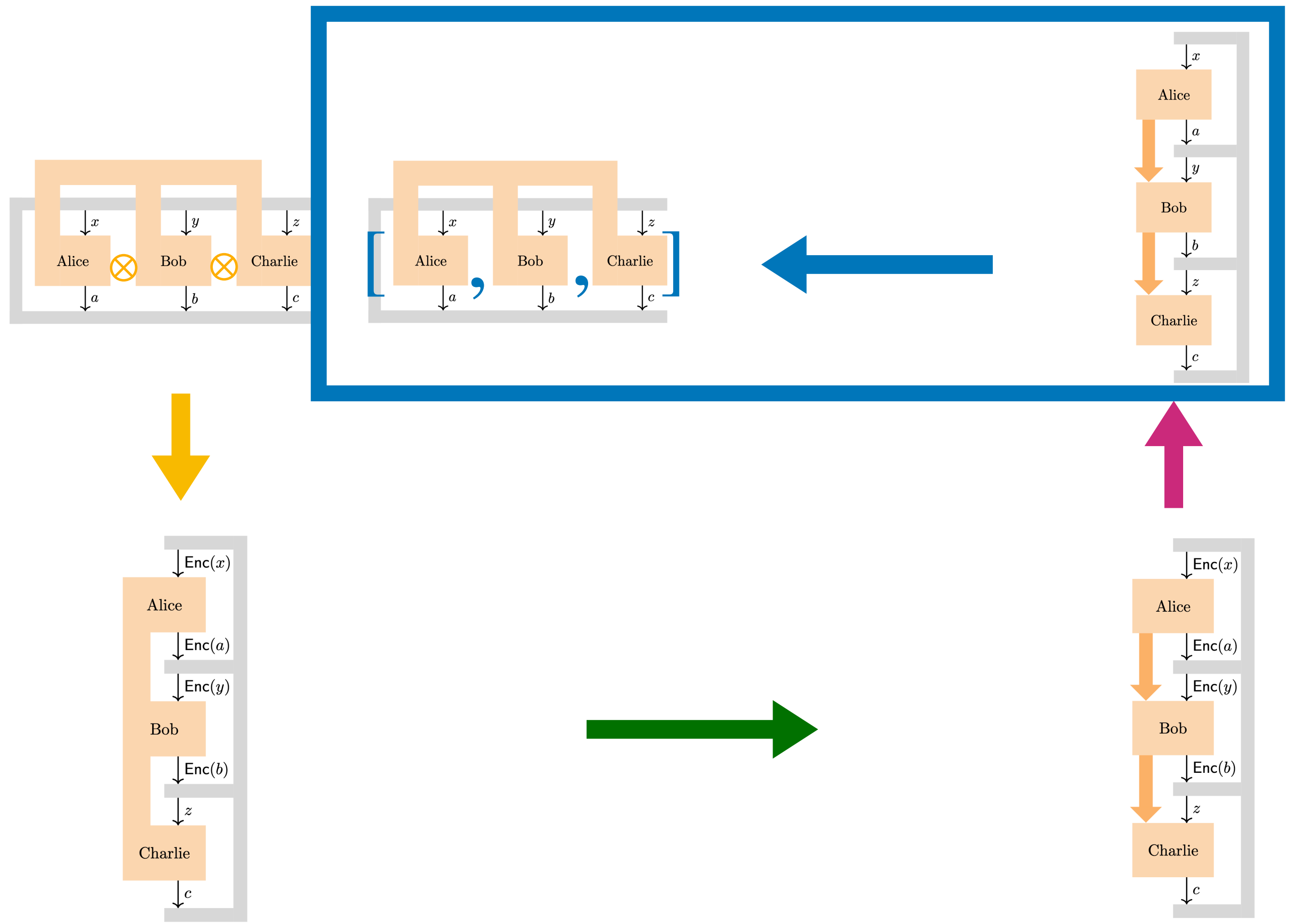
4. From sequential to non-local

Radon-Nikodym (RN) theorem



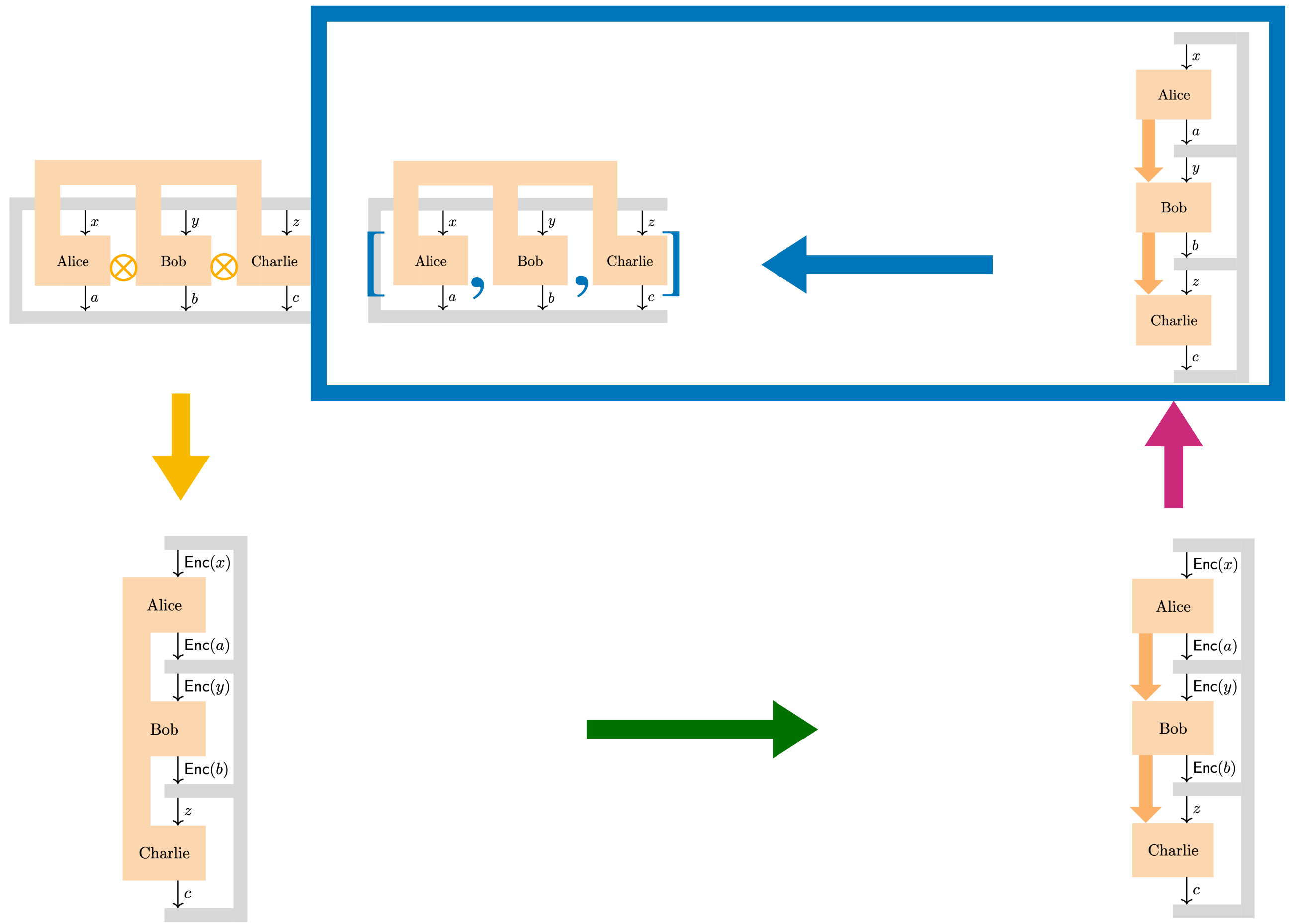
RN theorem for PL functionals

$$\psi(\mathfrak{a}) \leq \phi(\mathfrak{a}) \quad \Leftrightarrow \quad \begin{array}{l} \psi(\mathfrak{a}) = \langle \Omega_\phi | \, D_\psi \pi_\phi(\mathfrak{a}) \, | \Omega_\phi \rangle \\ [D_\psi, \pi_\phi(\mathscr{A})] = 0 \end{array}$$

The **GNS construction** of the dominant functional can be used to represent the dominated.

4. From sequential to non-local

Radon-Nikodym (RN) theorem



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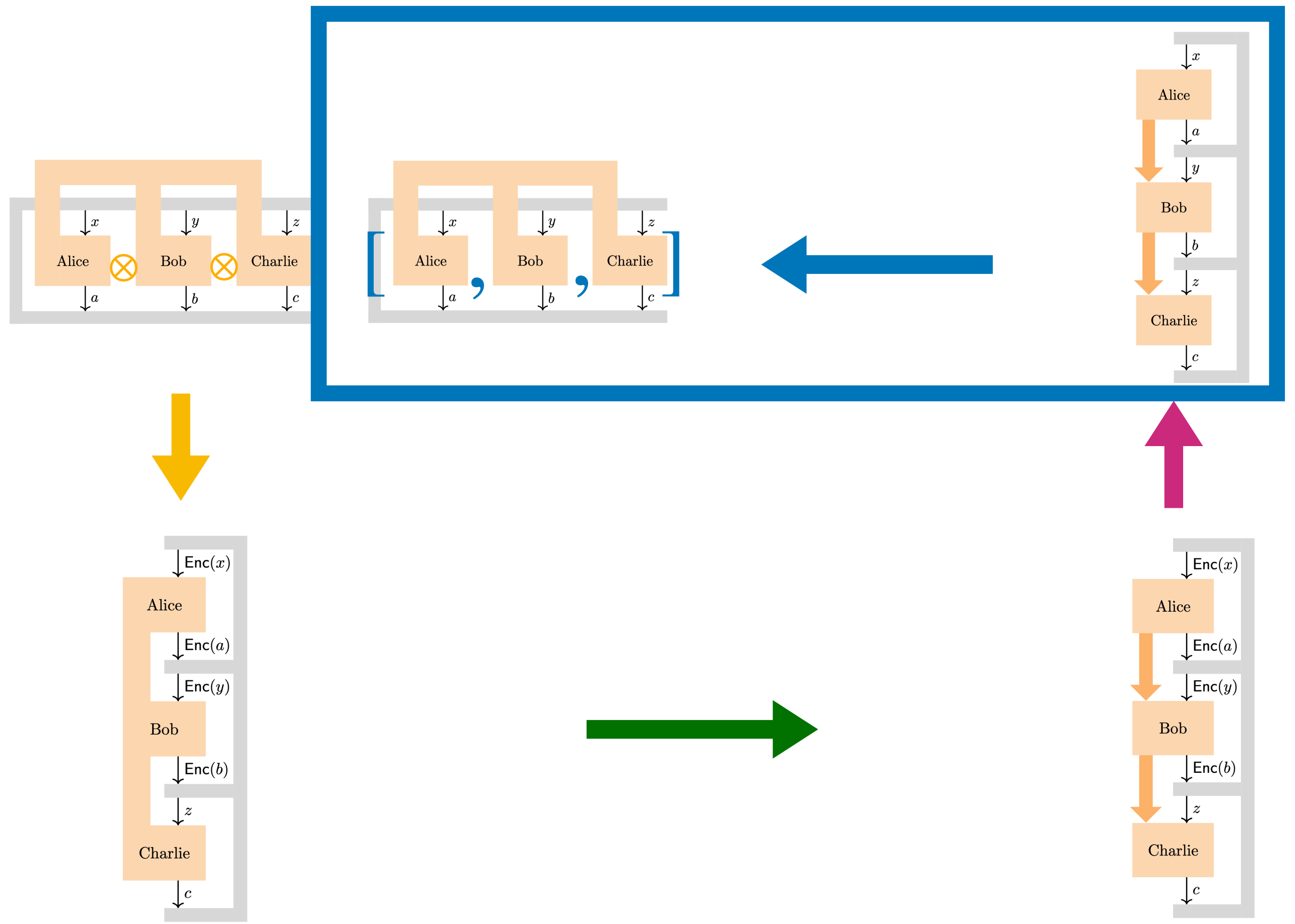
RN theorem for CP maps (adapted)

$$\sum_b T_{b|y}(\mathfrak{b}) = T(\mathfrak{b}) \quad \forall y \qquad \Leftrightarrow \qquad \begin{aligned} &T_{b|y}(\mathfrak{b}) = V_T^* D_{b|y} \pi_T(\mathfrak{b}) V_T \\ &[D_{b|y}, \pi_T(\mathfrak{b})] = 0 \qquad \forall \mathfrak{b} \in \mathscr{B} \end{aligned}$$

The **Stinespring dilation** of the dominant map can be used to represent the dominated.

4. From sequential to non-local

Radon-Nikodym (RN) theorem



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RN theorem for CP maps (adapted)

Stinespring isometry

$$\sum_b T_{b|y}(\mathfrak{b}) = T(\mathfrak{b}) \quad \forall y \qquad \Leftrightarrow \qquad \begin{array}{l} T_{b|y}(\mathfrak{b}) = V_T^* D_{b|y} \pi_T(\mathfrak{b}) V_T \\ [D_{b|y}, \pi_T(\mathfrak{b})] = 0 \qquad \forall \mathfrak{b} \in \mathscr{B} \end{array}$$

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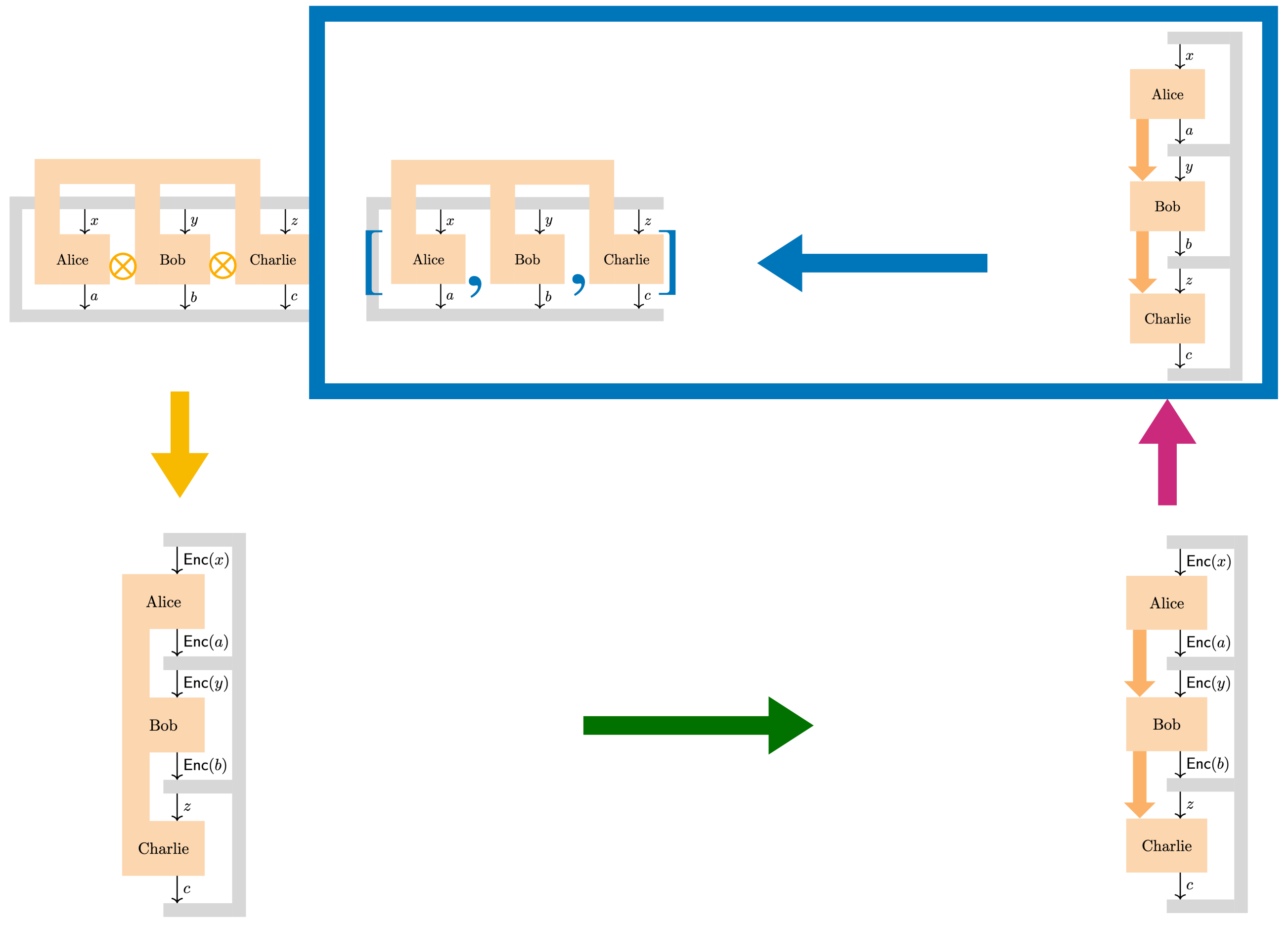
4. From sequential to non-local

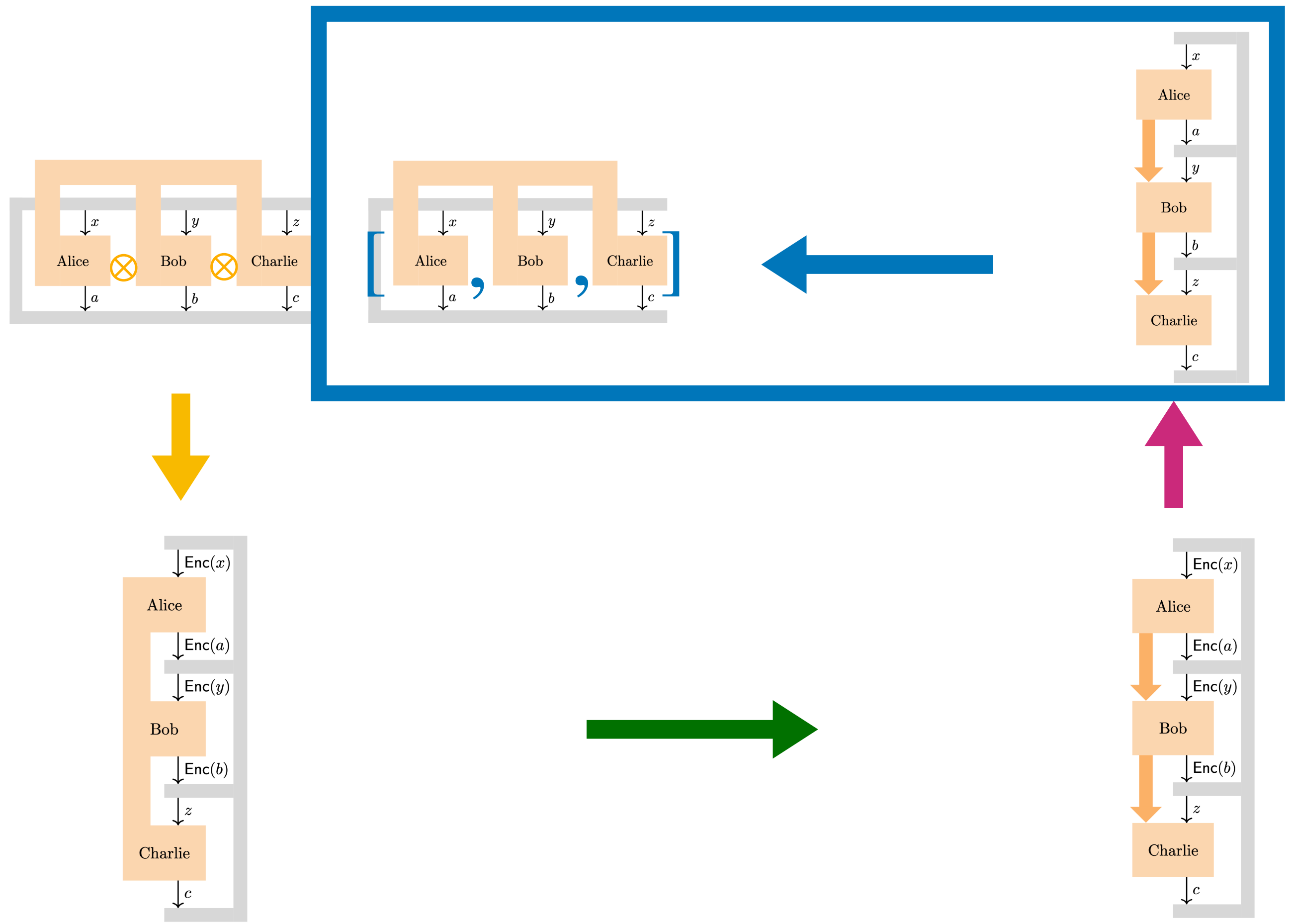
Radon-Nikodym theorem for CP maps

3 players Also transformation equivalences !



$$\phi_{a|x}(T_{b|y}(\mathfrak{m}_{c|z}))$$

4. From sequential to non-local



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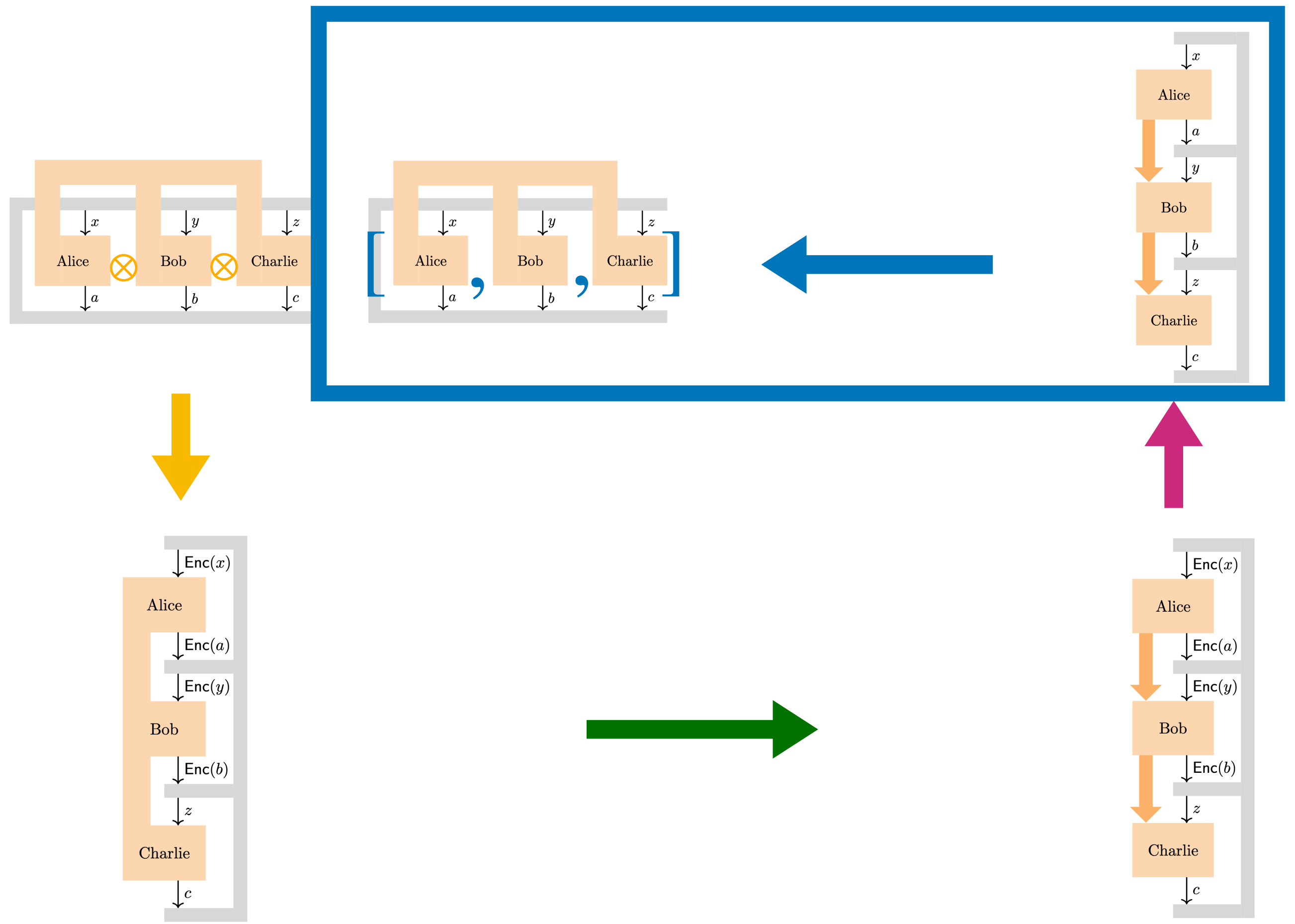

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$$\phi_{a|x}(T_{b|y}(\mathfrak{m}_{c|z})) = \langle \Omega_{\phi\circ T} | \, D_{ab|xy} \pi_{\phi\circ T}(\mathfrak{m}_{c|z}) \, | \Omega_{\phi\circ T} \rangle$$

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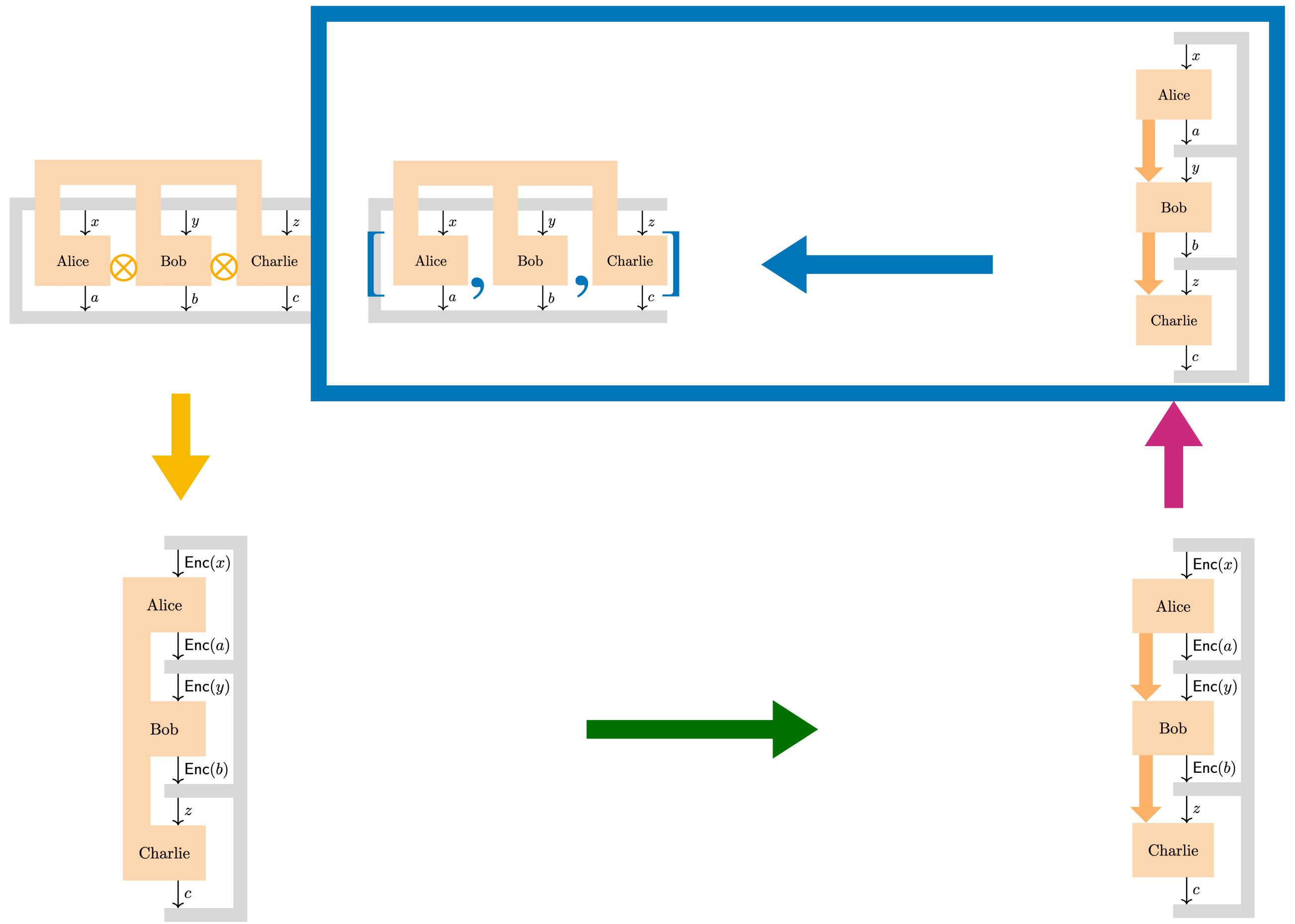

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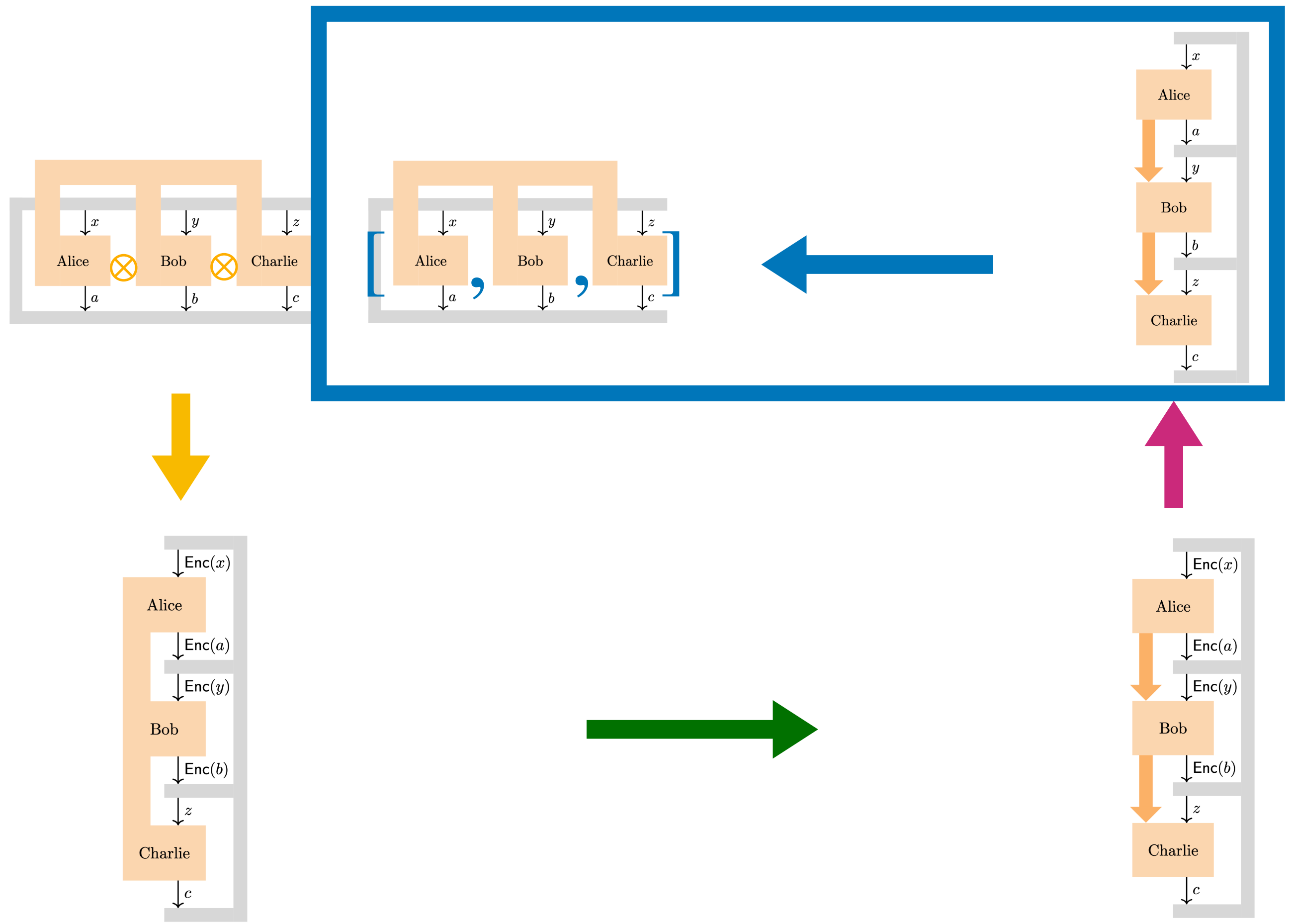

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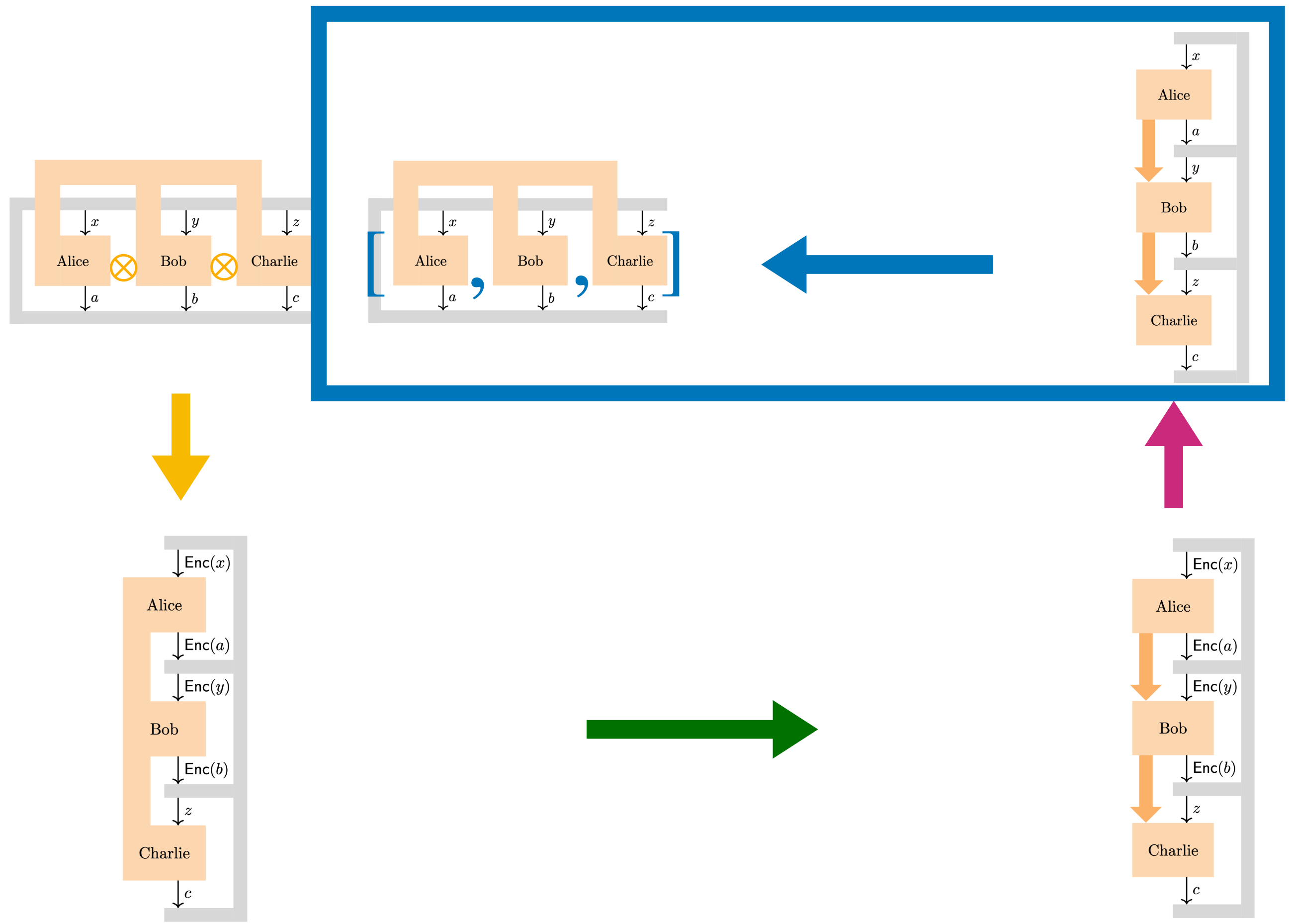

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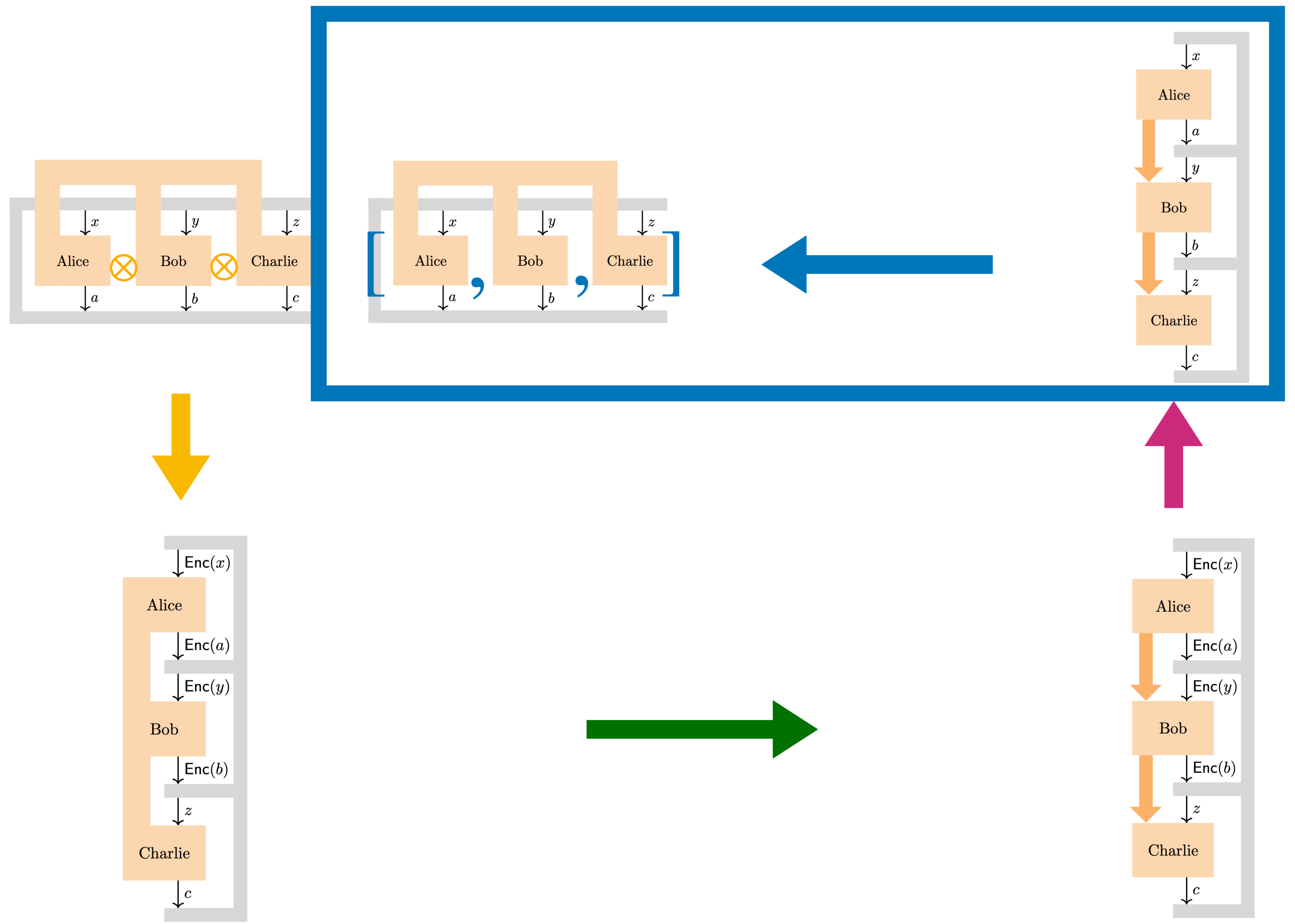

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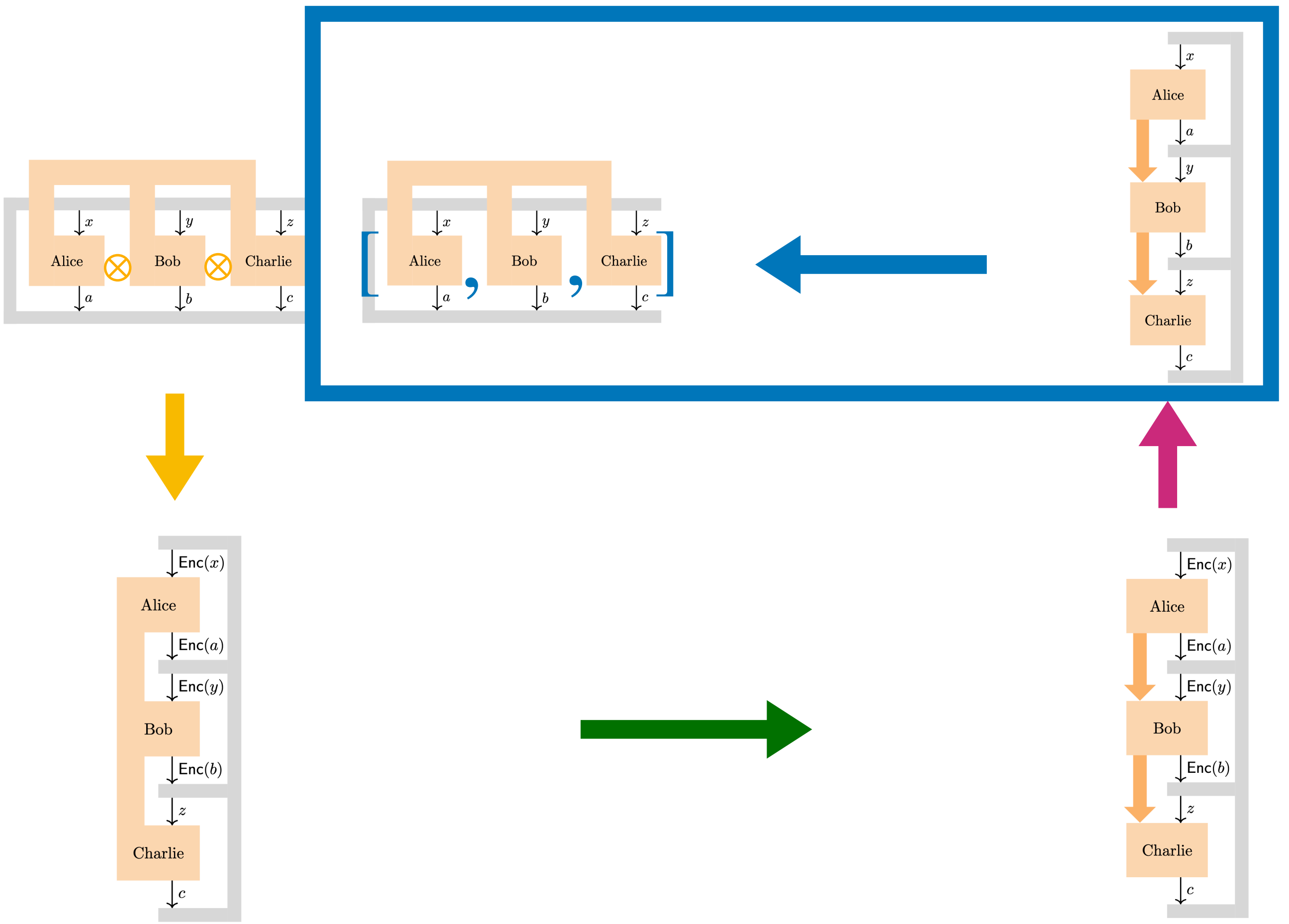

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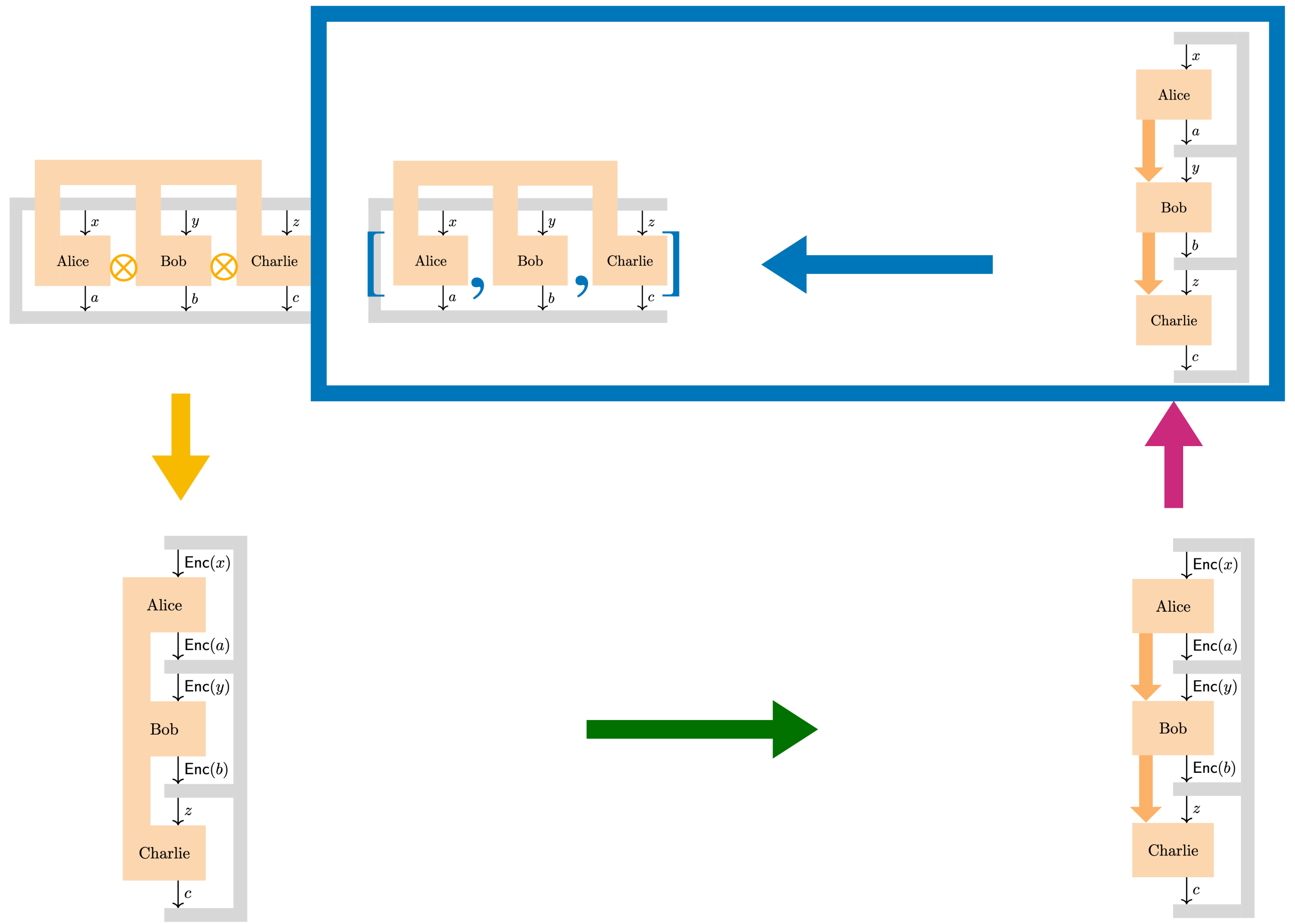

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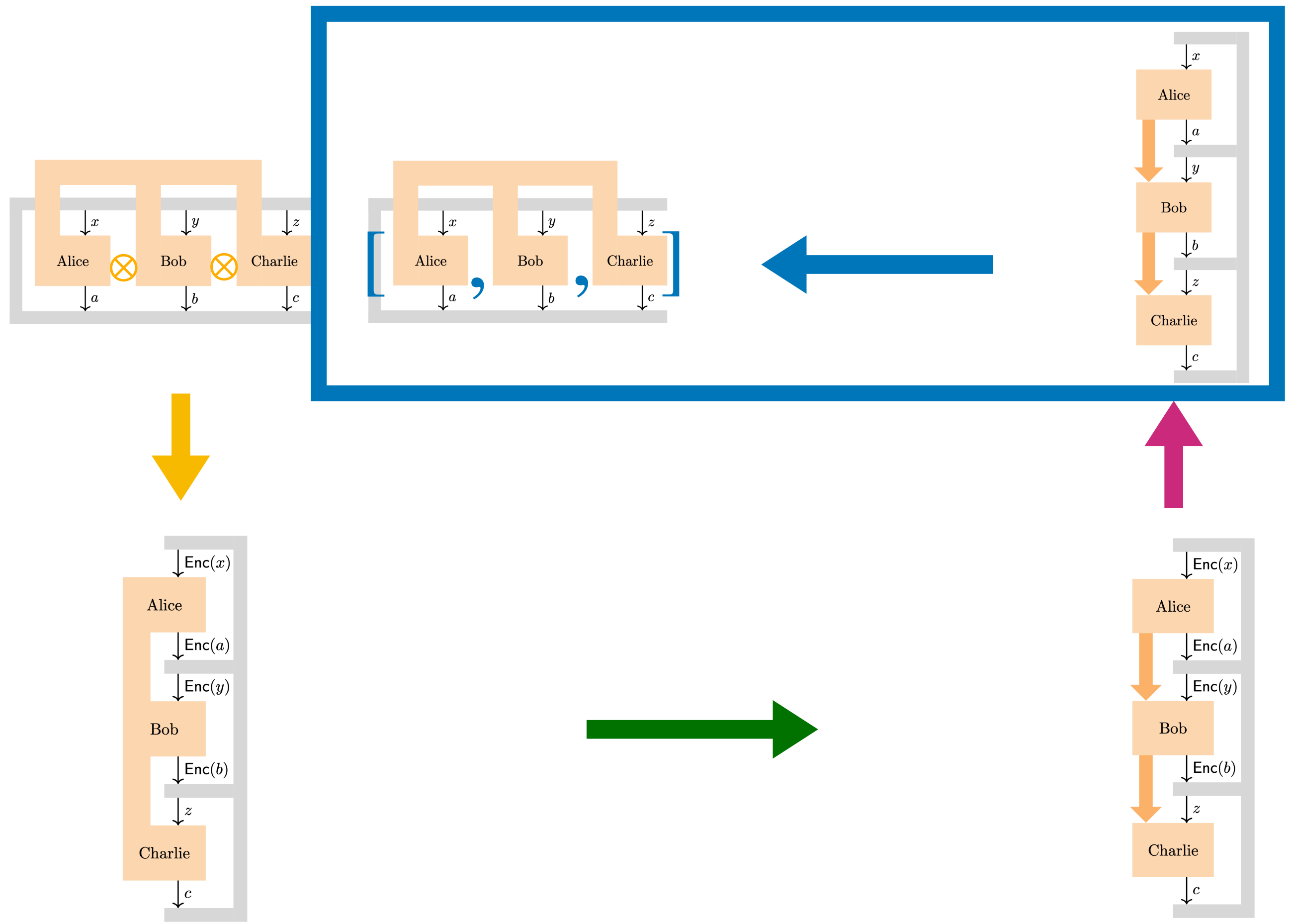

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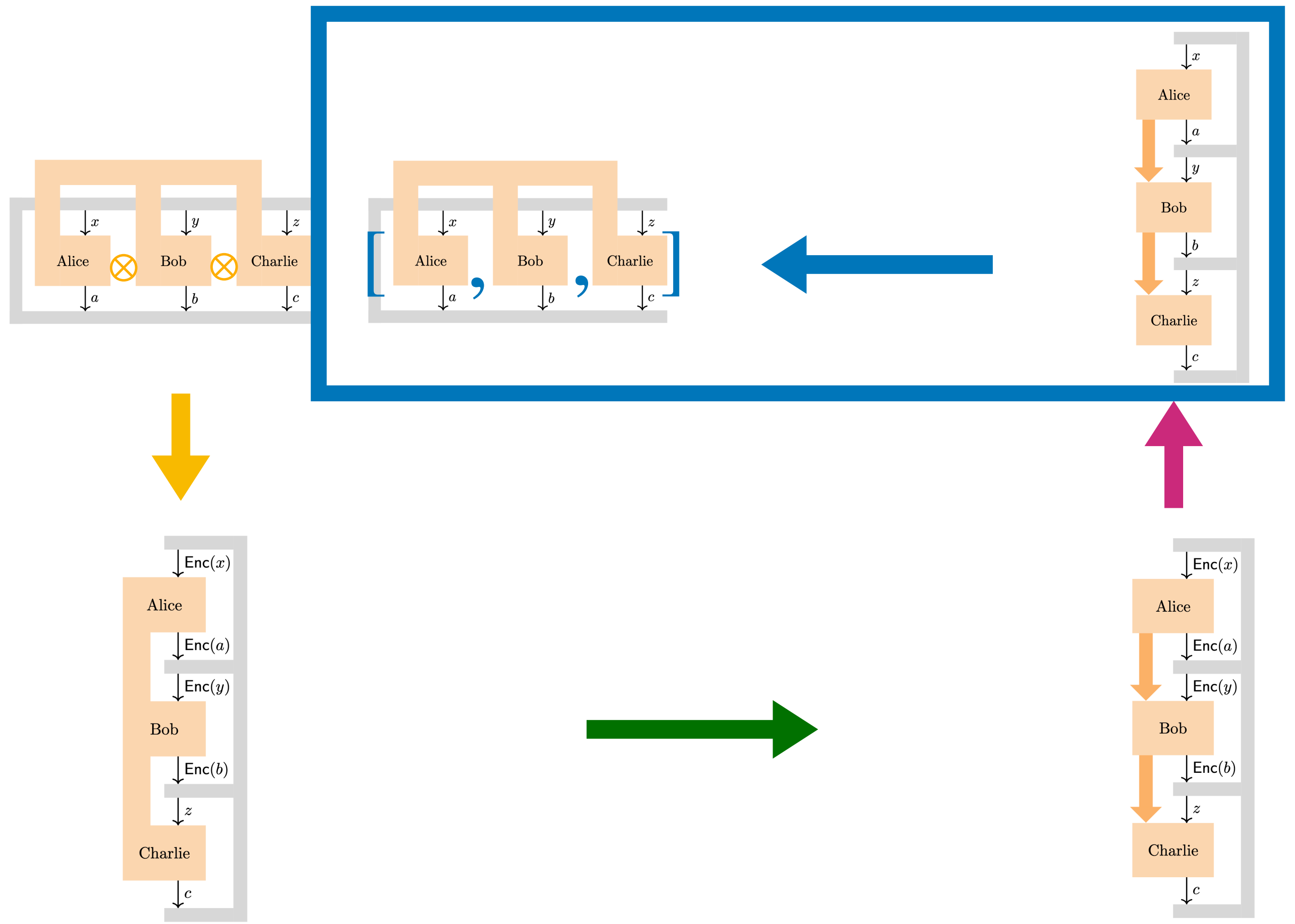
4. From sequential to non-local

Radon-Nikodym (RN) theorem



Chain rule of RN (adapted)

$$\sum_a \phi_{a|x}(\mathfrak{a}) = \phi(\mathfrak{a}) \quad \forall x \qquad \Leftrightarrow \qquad \phi_{a|x}(T_{b|y}(\mathfrak{m}_{c|z})) = \langle\Omega|\, \bar{D}_{a|x}\bar{D}_{b|y}\pi(\mathfrak{m}_{c|z})\,|\Omega\rangle$$

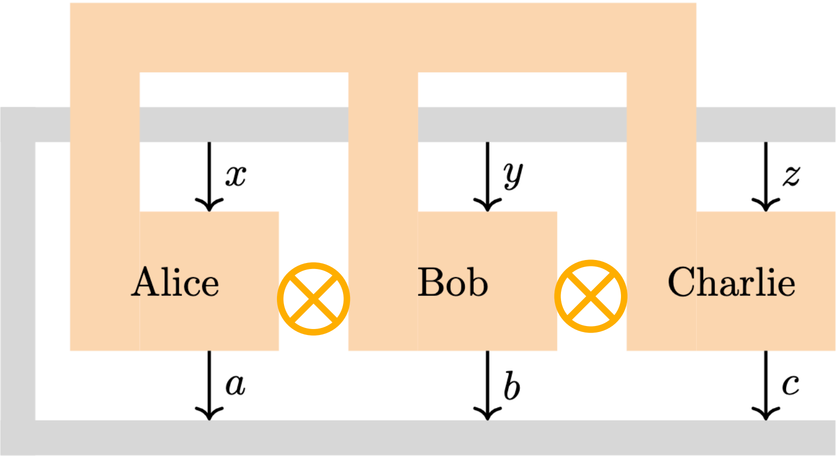
$$\sum_b T_{b|y}(\mathfrak{b}) = T(\mathfrak{b}) \quad \forall y$$

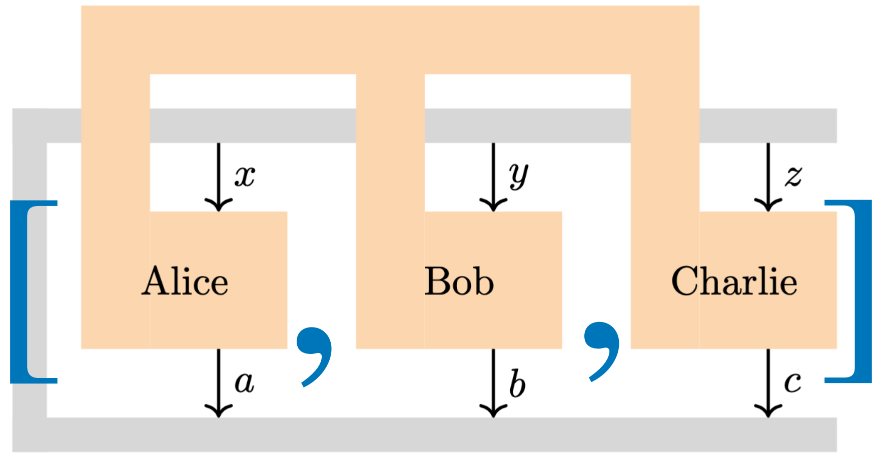
$$[\bar{D}_{a|x}, \bar{D}_{b|y}] = 0$$

$$[\bar{D}_{a|x}, \pi(\mathfrak{m}_{c|z})] = 0$$

$$[\bar{D}_{b|y}, \pi(\mathfrak{m}_{c|z})] = 0$$

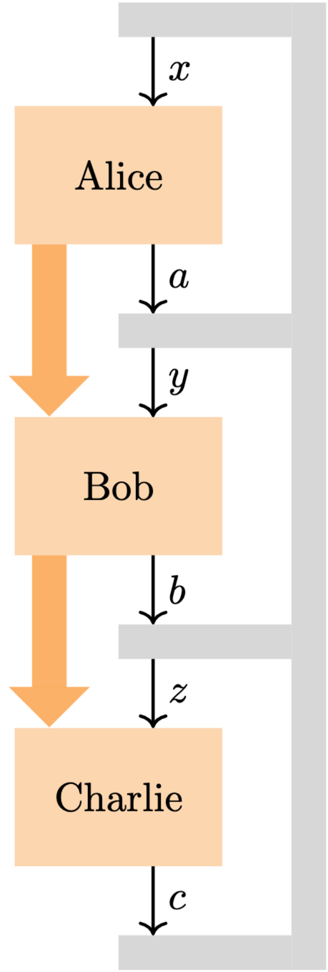
To sum up





Non-local game

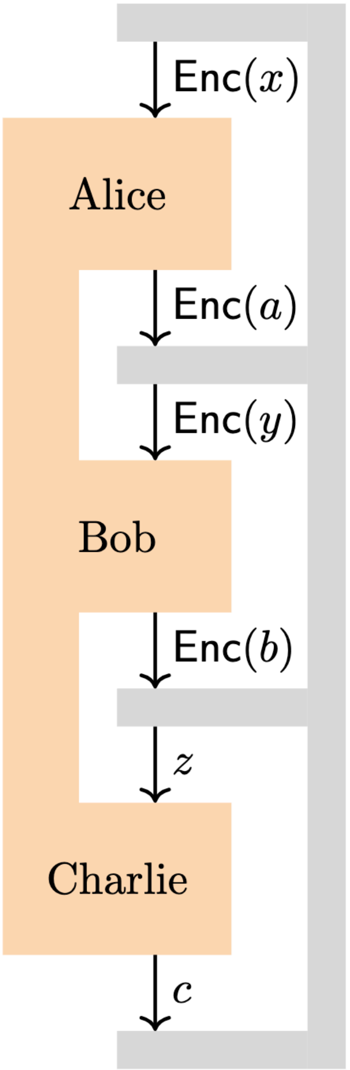
New chain rule for RN



+ Preparation equivalences

+ ~ Transformation equivalences

KLVY compiler

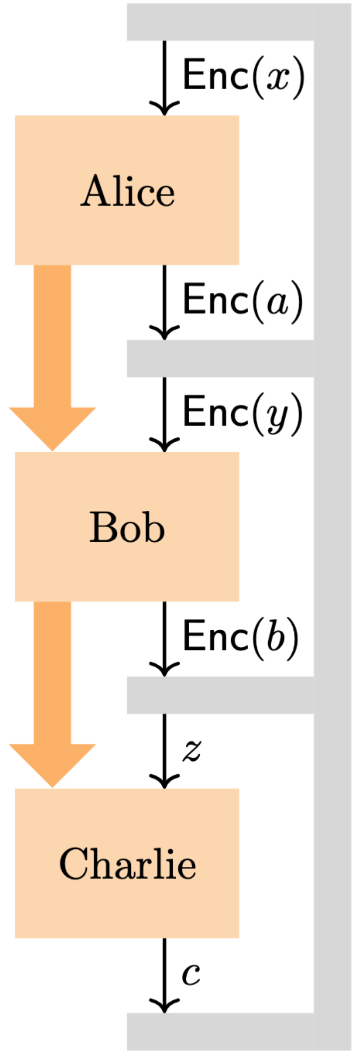


Sequential game

New block-encoding theorem

$\lambda \to \infty$

Universal C*algebra of sequential PVMs



+ Stronger constraints from IND-CPA

Conclusions

Asymptotic quantum soundness of the KLVY compiler for all multipartite games

Many new techniques to characterize q-instruments

Conclusions

- Asymptotic quantum soundness of the KLVY compiler for all multipartite games
- Many new techniques to characterize q-instruments
- Is commuting operator the tightest bound we can get?
- Convergence speed for finite levels of security?


Quantitative
quantum soundness
for all
multipartite games?

Thanks for listening !

References

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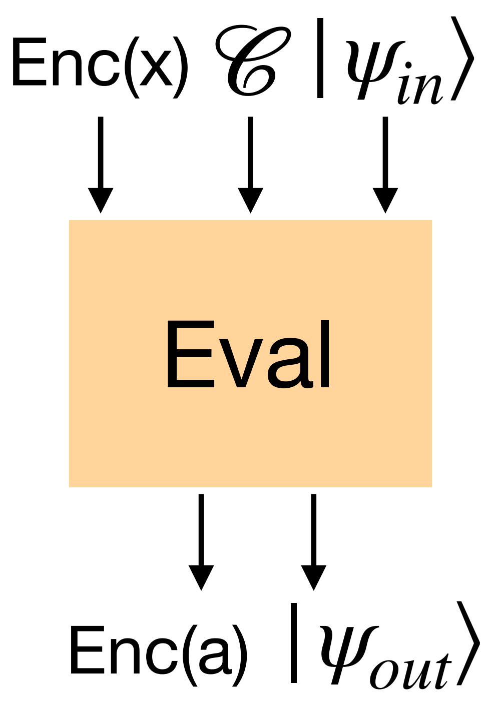
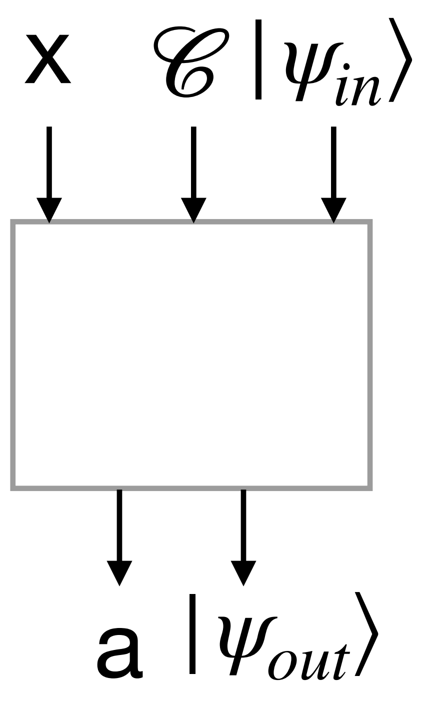
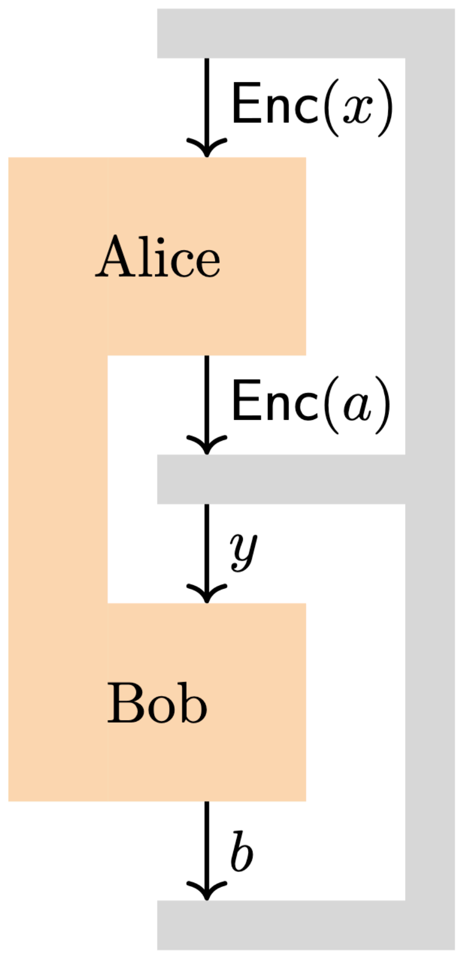
I. Klep, C. Paddock, M.O. Renou, S. Schmidt, L. Tendick, X. Xu, Y. Zhao

Bounding the asymptotic quantum value of all multipartite compiled nonlocal games.
M. Baroni, D. Leichtle, S. Janković, I. Šupić
arXiv: 2507.12408

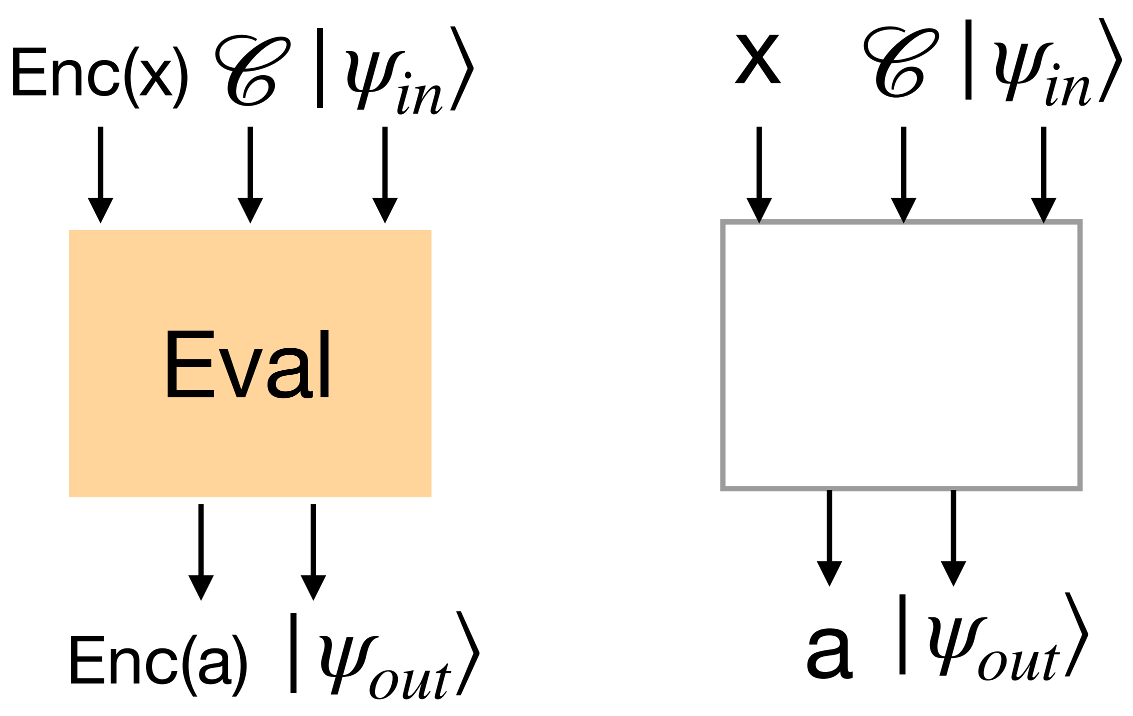
KLVY compiler : QFHE

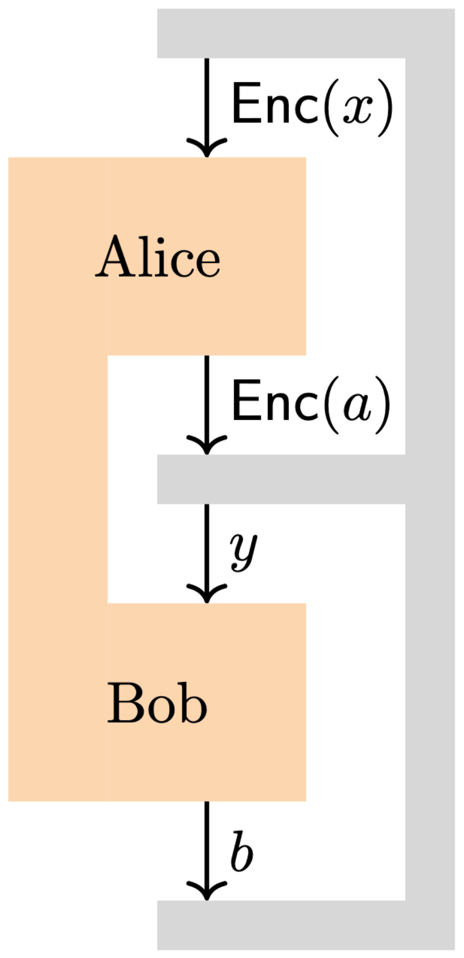






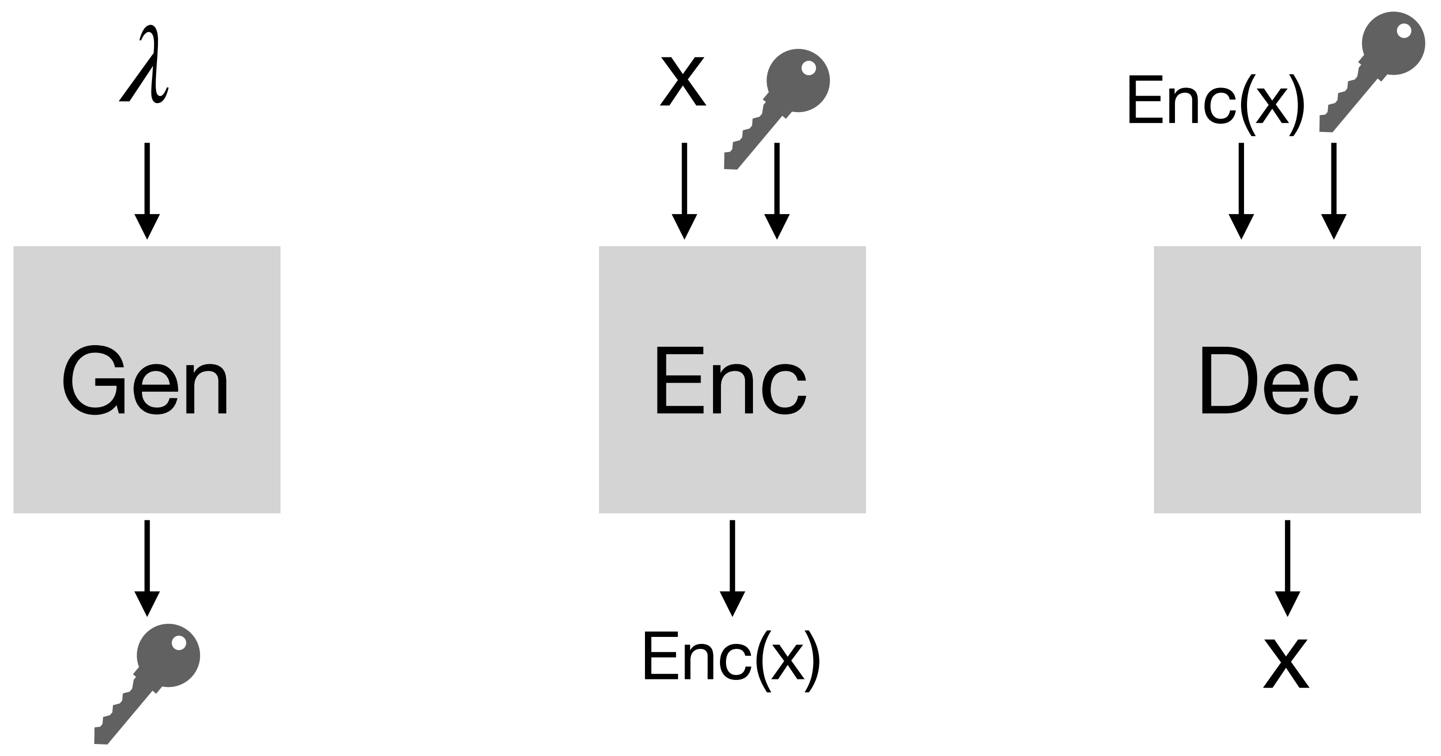
KLVY compiler : QFHE

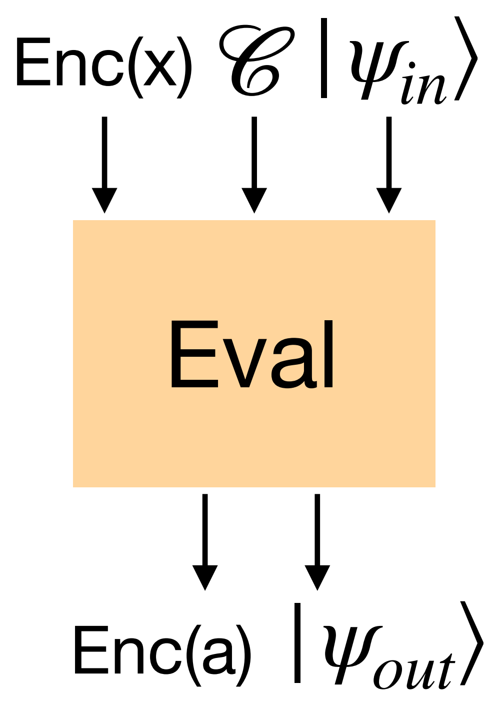


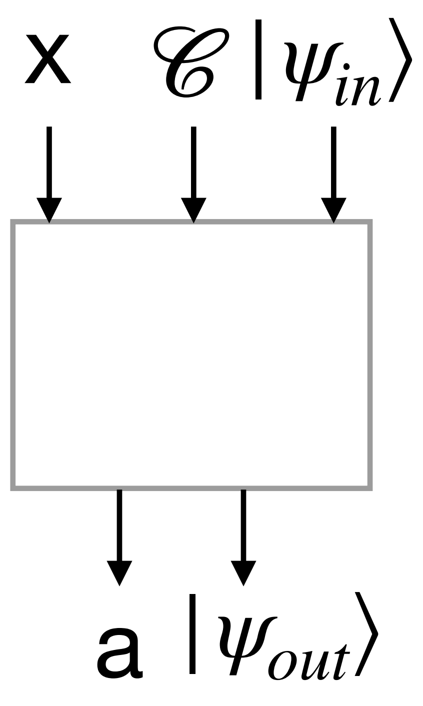


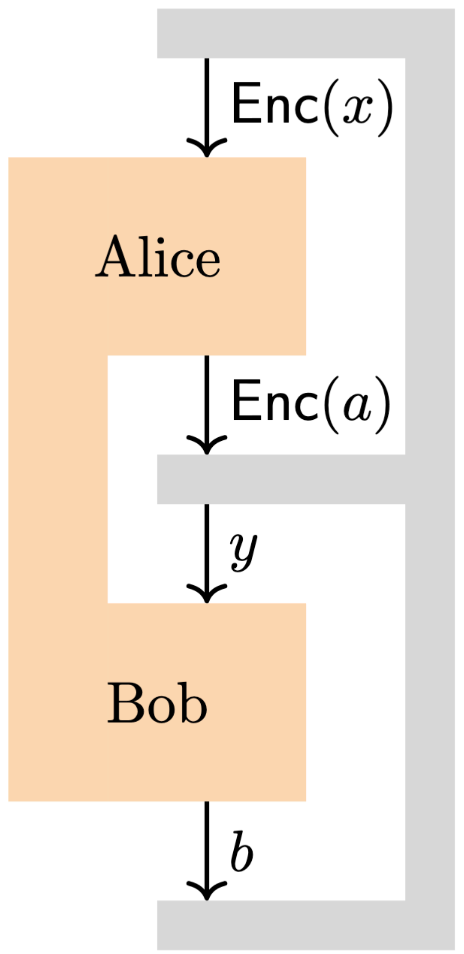
Quantum Fully Homomorphic Encryption scheme (QFHE) with

KLVY compiler : QFHE



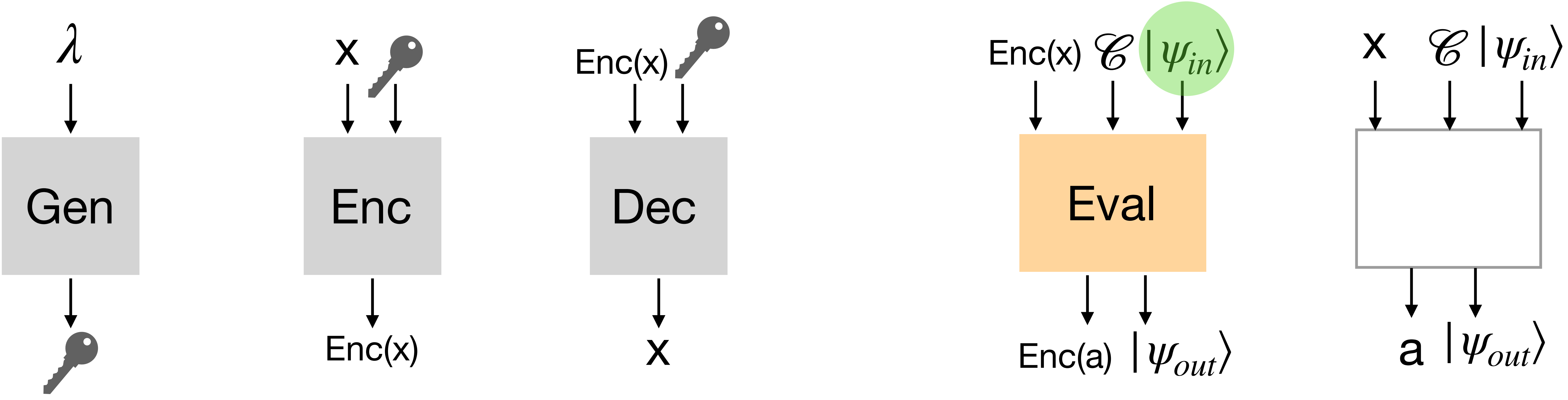


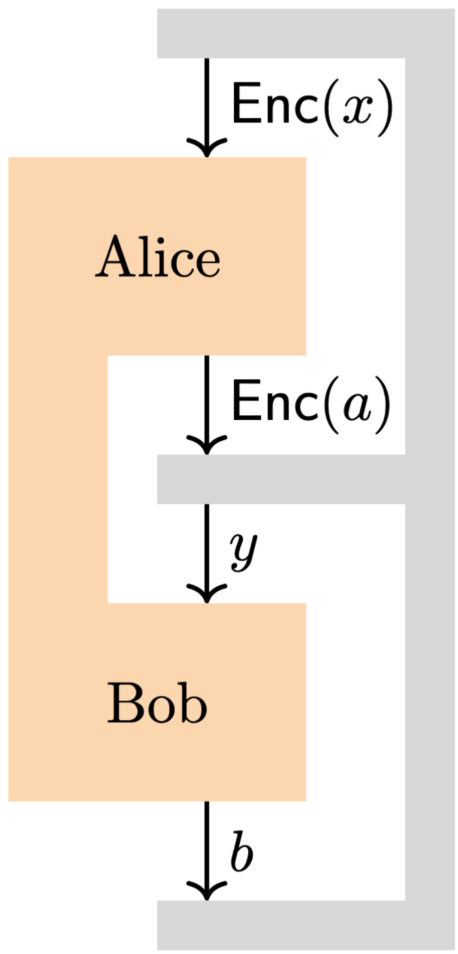




Quantum Fully Homomorphic Encryption scheme (QFHE) with

KLVY compiler : QFHE

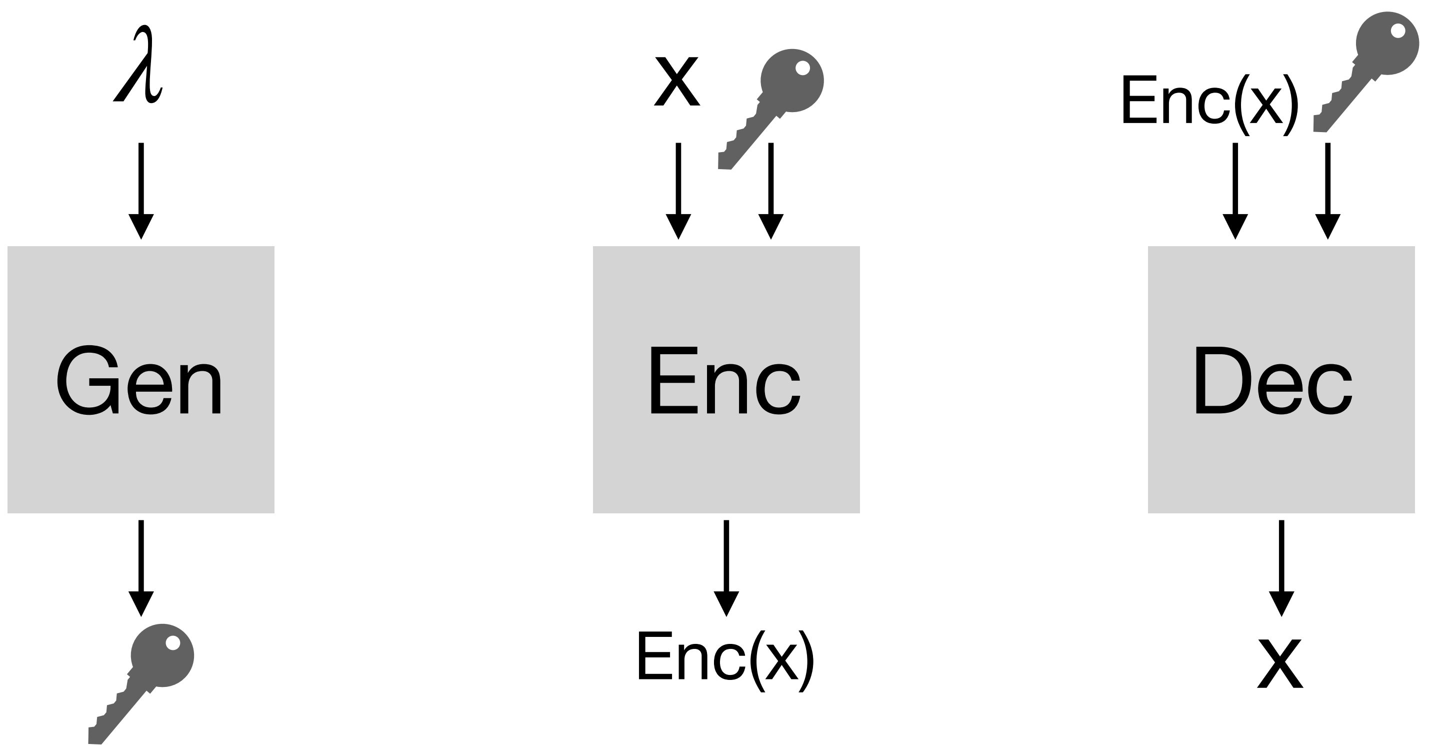


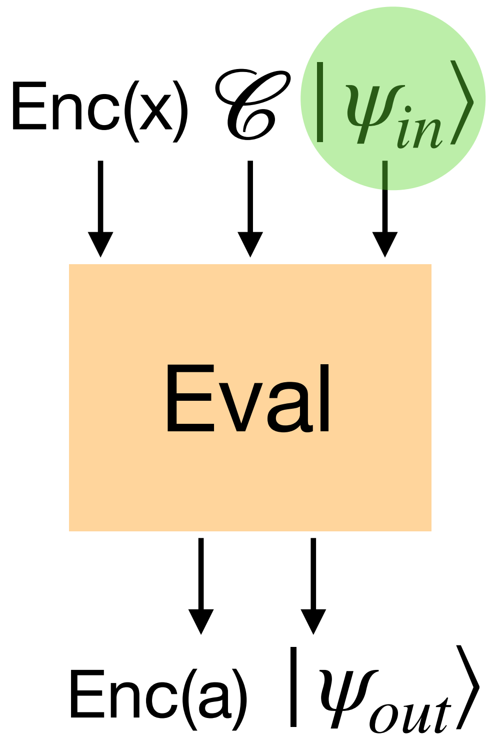


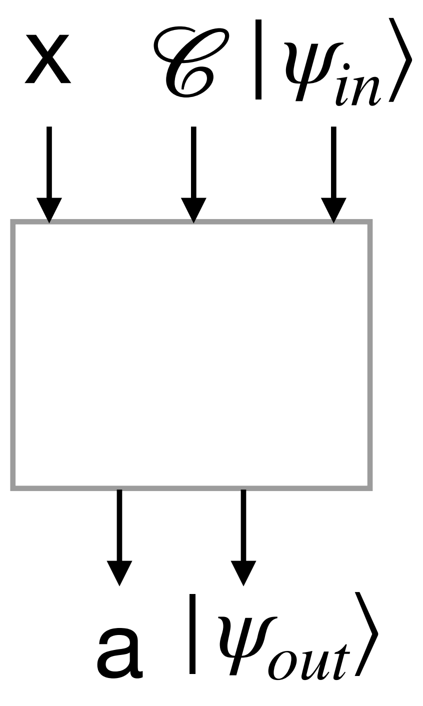
Quantum Fully Homomorphic Encryption scheme (QFHE) with

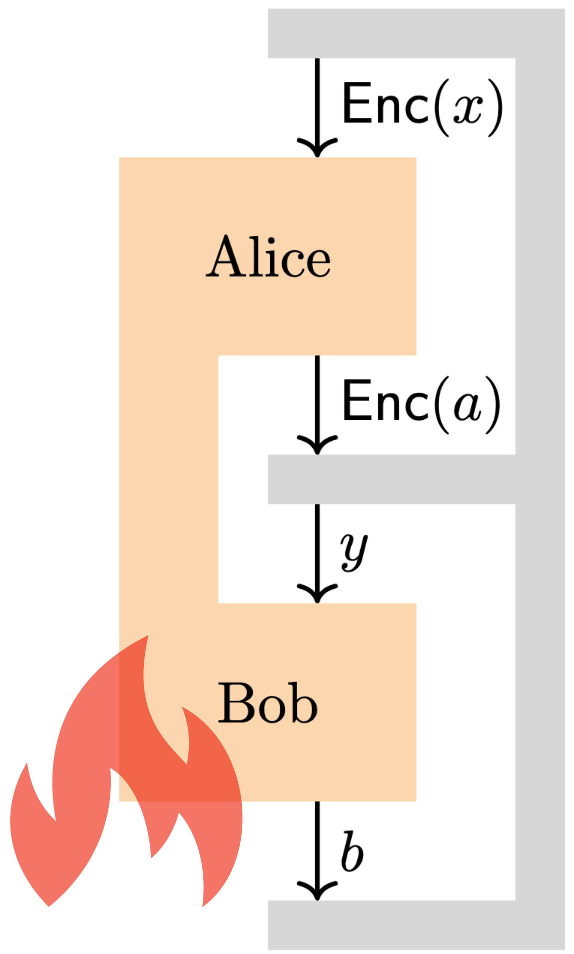
- correctness with auxiliary input

KLVY compiler : QFHE





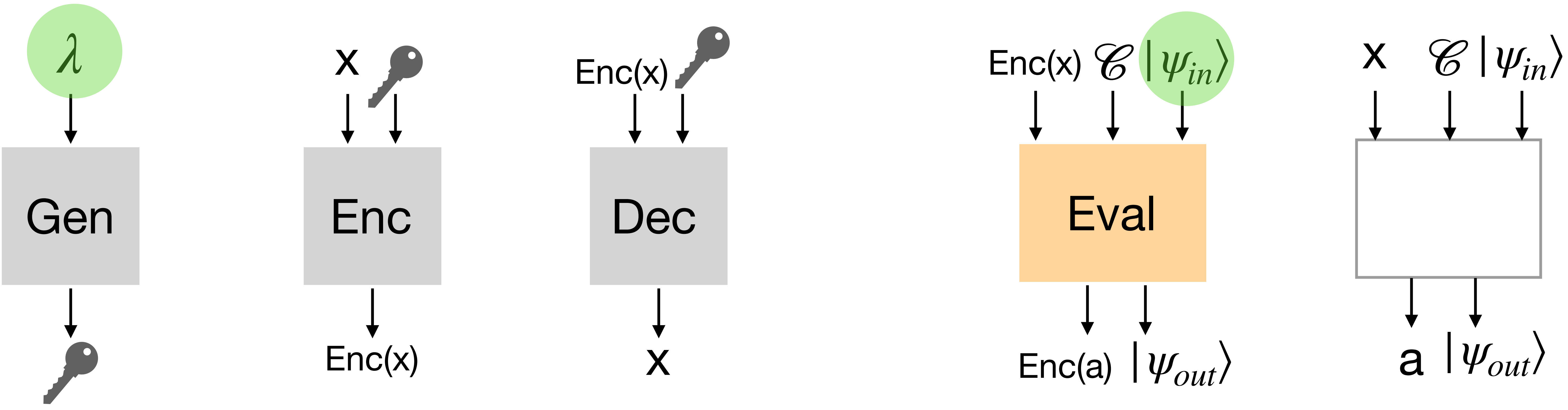


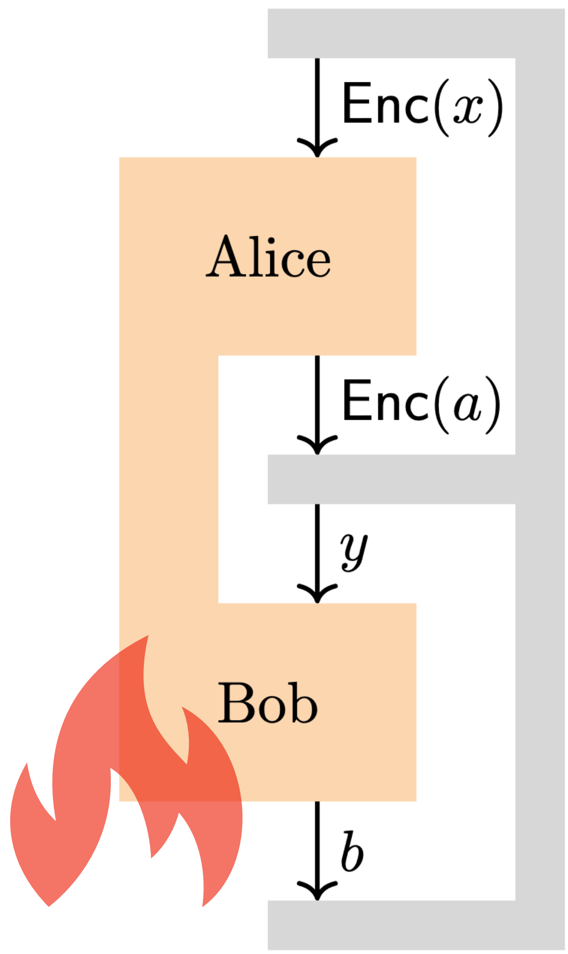


Quantum Fully Homomorphic Encryption scheme (QFHE) with

- correctness with auxiliary input
- IND-CPA security against QPT adversaries

KLVY compiler : QFHE





Quantum Fully Homomorphic Encryption scheme (QFHE) with

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